Stability and stabilization of linear switched systems in finite and infinite dimensions

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Stabilité et stabilisation de systèmes linéaires à commutation en dimensions finie et infinie

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Stabilité et stabilisation de systèmes linéaires à commutation en dimensions finie et infinie

Motivée par des travaux précédents sur la stabilisation de systèmes à excitation persistante, cette thèse s'intéresse à la stabilité et à la stabilisation de systèmes linéaires à commutation en dimensions finie et infinie. Après une introduction générale présentant les principales motivations et les résultats importants de la littérature, on aborde quatre sujets.

On commence par l'étude d'un système linéaire en dimension finie à commutation aléatoire. Le temps passé en chaque sous-système i est choisi selon une loi de probabilité ne dépendant que de i, les commutations entre sous-systèmes étant déterminées par une chaine de Markov discrète. On caractérise les exposants de Lyapunov en appliquant le Théorème ergodique multiplicatif d'Oseledets à un système associé en temps discret, et on donne une expression pour l'exposant de Lyapunov maximal. Ces résultats sont appliqués à un système de contrôle à commutation. Sous une hypothèse de contrôlabilité, on montre que ce système peut être stabilisé presque surement avec taux de convergence arbitraire, ce qui est en contraste avec les systèmes déterministes à excitation persistante.

On considère ensuite un système de N équations de transport avec amortissement interne à excitation persistante, couplées linéairement par le bord à travers une matrice M, ce qui peut être vu comme un système d'EDPs sur un réseau étoilé. On montre que, si l'activité de l'amortissement intermittent est déterminée par des signaux à excitation persistante, alors, sous des bonnes hypothèses sur M et sur la rationalité des rapports entre les longueurs des arêtes du réseau, ce système est exponentiellement stable, uniformément par rapport aux signaux à excitation persistante. Ce résultat est montré grâce à une formule explicite pour les solutions du système, qui permet de bien suivre les effets de l'amortissement intermittent.

Le sujet suivant que l'on considère est le comportement asymptotique d'équations aux différences non-autonomes. On obtient une formule explicite pour les solutions en termes des conditions initiales et de certains coefficients matriciels dépendants du temps, qui généralise la formule obtenue pour le système de N équations de transport. Le comportement asymptotique des solutions est caractérisé à travers les coefficients matriciels. Dans le cas d'équations aux différences à commutation arbitraire, on obtient un résultat de stabilité qui généralise le critère de Hale–Silkowski pour les systèmes autonomes. Grâce à des transformations classiques d'EDPs hyperboliques en équations aux différences, on applique ces résultats au transport et à la propagation d'ondes sur des réseaux.

Finalement, la formule explicite précédente est généralisée à une équation aux différences contrôlée, dont la contrôlabilité est alors analysée. La contrôlabilité relative est caractérisée à travers un critère algébrique sur les coefficients matriciels de la formule explicite, ce qui généralise le critère de Kalman. On compare également la contrôlabilité relative pour des retards différents en termes de leur structure de dépendance rationnelle, et on donne une borne sur le temps minimal de contrôlabilité. Pour des systèmes avec retards commensurables, on montre que la contrôlabilité exacte est équivalente à l'approchée et on donne un critère qui les caractérise. On analyse également la contrôlabilité exacte et approchée de systèmes en dimension 2 avec deux retards sans l'hypothèse de commensurabilité.

Mots-clés. Systèmes à commutation, stabilité, stabilisation, excitation persistante, exposants de Lyapunov, commutation aléatoire, équation de transport, équation des ondes, équations aux différences, contrôlabilité, réseaux.

2010 Mathematics Subject Classification. 93C30, 93D05, 93D15, 39A30, 35B35, 35R02, 37H15.

Stability and stabilization of linear switched systems in finite and infinite dimensions

Motivated by previous work on the stabilization of persistently excited systems, this thesis addresses stability and stabilization issues for linear switched systems in finite and infinite dimensions. After a general introduction presenting the main motivations and important results from the literature, we analyze four problems.

The first system we study is a linear finite-dimensional random switched system. The time spend on each subsystem i is chosen according to a probability law depending only on i, and the switches between subsystems are determined by a discrete Markov chain. We characterize the Lyapunov exponents by applying Oseledets' Multiplicative Ergodic Theorem to an associated discrete-time system, and provide an expression for the maximal Lyapunov exponent. These results are applied to a switched control system, showing that, under a controllability hypothesis, almost sure stabilization can be achieved with arbitrarily large decay rates, a situation in contrast to deterministic persistently excited systems.

We next consider a system of N transport equations with intermittent internal damping, linearly coupled by their boundary conditions through a matrix M, which can be seen as a system of PDEs on a star-shaped network. We prove that, if the activity of the intermittent damping terms is determined by persistently exciting signals, then, under suitable hypotheses on M and on the rationality of the ratios between the lengths of the network edges, such system is exponentially stable, uniformly with respect to the persistently exciting signals. The proof of this result is based on an explicit representation formula for the solutions of the system, which allows one to efficiently track down the effects of the intermittent damping.

The following topic we address is the asymptotic behavior of non-autonomous difference equations. We obtain an explicit representation formula for their solutions in terms of their initial conditions and some time-dependent matrix coefficients, which generalizes the one for the system of N transport equations. The asymptotic behavior of solutions is characterized in terms of the matrix coefficients. In the case of difference equations with arbitrary switching, we obtain a stability result which generalizes Hale–Silkowski criterion for autonomous systems. Using classical transformations of hyperbolic PDEs into difference equations, we apply our results to transport and wave propagation on networks.

Finally, we generalize the previous representation formula to a controlled difference equation, whose controllability is then analyzed. Relative controllability is characterized in terms of an algebraic property on the matrix coefficients from the explicit formula, generalizing Kalman criterion. We also compare the relative controllability for different delays in terms of their rational dependence structure, and provide a bound on the minimal controllability time. Exact and approximate controllability for systems with commensurable delays are characterized and proved to be equivalent. We also describe exact and approximate controllability for two-dimensional systems with two delays not necessarily commensurable.

Keywords. Switched systems, stability, stabilization, persistent excitation, Lyapunov exponents, random switching, transport equation, wave equation, difference equations, controllability, networks.

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Estabilidade e estabilização de sistemas chaveados lineares em dimensões finita e infinita

Esta tese, motivada por trabalhos anteriores sobre a estabilização de sistemas a excitação persistente, se interessa à estabilidade e à estabilização de sistemas lineares chaveados em dimensões finita e infinita. Após uma introdução geral apresentando as principais motivações e os resultados importantes da literatura, quatro problemas são analisados.

O primeiro sistema estudado é um sistema chaveado aleatório linear em dimensão finita. O tempo passado em cada sub-sistema i é escolhido segundo uma lei de probabilidade dependente apenas de i, e as comutações entre os sub-sistemas provêm de uma cadeia de Markov discreta. Os expoentes de Lyapunov do sistema são caracterizados através da aplicação do Teorema Ergódico Multiplicativo de Oseledets a um sistema a tempo discreto associado, obtendo-se igualmente uma expressão para o expoente de Lyapunov máximo. Os resultados são aplicados a um sistema de controle chaveado, mostrando que, sob uma hipótese de controlabilidade, pode-se efetuar estabilização quase certa com taxas de convergência arbitrárias, uma situação em contraste com sistemas a excitação persistente deterministas.

Em seguida, considera-se um sistema de N equações de transporte com amortecimento interno intermitente, acopladas linearmente pela fronteira através de uma matriz M. Este sistema pode ser visto como um sistema de EDPs numa rede estrelada. Mostra-se que, se a atividade dos termos de amortecimento intermitentes for determinada por sinais a excitação persistente, então, sob hipóteses adequadas sobre M e sobre a racionalidade das razões entre os comprimentos das arestas da rede, o sistema é exponencialmente estável, uniformemente em relação aos sinais a excitação persistente. A demonstração deste resultado se baseia numa fórmula de representação explícita das soluções do sistema, permitindo analisar claramente os efeitos do amortecimento intermitente.

O tópico seguinte a ser considerado é o comportamento assintótico de equações a diferenças não-autônomas. Obtém-se uma fórmula explícita para suas soluções em função das condições iniciais e de coeficientes matriciais dependentes do tempo, generalizando assim a fórmula explícita obtida para o sistema de N equações de transporte. O comportamento assintótico das soluções é caracterizado através dos coeficientes matriciais. No caso de equações a diferenças com chaveamento arbitrário, obtém-se um resultado de estabilização que generaliza o critério de Hale–Silkowski para sistemas autônomos. Transformações clássicas de EDPs hiperbólicas em equações a diferenças são usadas para se aplicar os resultados obtidos a equações de transporte e à propagação de ondas em redes.

Finalmente, a fórmula explícita anterior é generalizada a uma equação a diferenças controlada, cuja controlabilidade é então analisada. A controlabilidade relativa é caracterizada através de uma propriedade algébrica dos coeficientes matriciais da fórmula explícita, o que generaliza o critério de Kalman. Compara-se também a controlabilidade relativa para atrasos diferentes em termos de suas estruturas de dependência racional, e obtém-se um limite superior ao tempo mínimo de controlabilidade. Para sistemas com atrasos comensuráveis, mostra-se que a controlabilidade exata e a aproximada são equivalente, fornecendo um critério para ambas. Também se descreve a controlabilidade exata e a aproximada de sistemas bidimensionais com dois atrasos não necessariamente comensuráveis.

Palavras-chave. Sistemas chaveados, estabilidade, estabilização, excitação persistente, expoentes de Lyapunov, chaveamento aleatório, equação de transporte, equação de onda, equações de diferenças, controlabilidade, redes.

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Contents

Notations

$\mathbb{N}$, $\mathbb{N}^*$	Sets of non-negative and positive integers, respectively.
$\mathbb{R}_+$, $\mathbb{R}_+^*$	$[0,+\infty)$ and $(0,+\infty)$, respectively.
$[\![a,b]\!]$	$[a,b]\cap\mathbb{Z}$, where $a,b\in\mathbb{R}$ and $[a,b]=\emptyset$ if $a>b$.
$\underline{N}$	$[\![1,N]\!]$, where $N\in\mathbb{N}^*$.
$\overline{F}$	Closure of the subset F of a topological space.
$\overline{z}$	Complex conjugate of $z\in\mathbb{C}$.
$x+F$	$\{x+y \mid y\in F\}$ for $x\in\mathbb{R}$ and $F\subset\mathbb{R}$.
$\#F$	Cardinality of the set F.
δ_{ij}	Kronecker symbol of i,j.
$x_\pm$	$\max(\pm x,0)$ if $x\in\mathbb{R}$, extended componentwise to vectors $x\in\mathbb{R}^d$.
$x_{\min}$, $x_{\max}$	Smallest and largest components of the vector $x\in\mathbb{R}^d$, respectively.
$\lfloor x\rfloor$, $\lceil x\rceil$	Floor and ceiling functions, denoting the unique integers satisfying $x-1<\lfloor x\rfloor\leq x$ and $x\leq\lceil x\rceil<x+1$ for $x\in\mathbb{R}$.
$\{x\}_y$	$x-\lfloor x/y\rfloor y$ for $x\in\mathbb{R}$ and $y>0$. Written simply as $\{x\}$ when $y=1$ and there is no possibility of confusion with the set contaning only the point x.
$\binom{n}{m}$	$\dfrac{n!}{m!(n-m)!}$ for $n,m\in\mathbb{N}$ with $m\leq n$.
χ_A	Characteristic function of the set A.
$\log x$	Natural logarithm of $x\in\mathbb{R}_+^*$.
$\operatorname{Re} z$, $\operatorname{Im} z$	Real and imaginary parts of $z\in\mathbb{C}$, respectively.
$\mathcal{M}_{d,m}(K)$	Set of $d\times m$ matrices with coefficients in $K\subset\mathbb{C}$. The set $\mathcal{M}_{d,1}(\mathbb{R})$ is canonically identified with $\mathbb{R}^d$, and similarly for $\mathcal{M}_{d,1}(\mathbb{C})$.
$\mathcal{M}_d(K)$	$\mathcal{M}_{d,d}(K)$.
$\mathrm{GL}_d(\mathbb{K})$	General linear group in $\mathbb{K}^d$ for $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.
Id_d	Identity matrix in $\mathcal{M}_d(\mathbb{R})$ or $\mathcal{M}_d(\mathbb{C})$.
$0_{d,m}$	Zero matrix in $\mathcal{M}_{d,m}(\mathbb{R})$ or $\mathcal{M}_{d,m}(\mathbb{C})$, denoted simply by 0 when d and m are clear from the context.
$\operatorname{diag}(a_1,\dots,a_d)$	Diagonal matrix in $\mathcal{M}_d(\mathbb{C})$ whose diagonal elements are $a_1,\dots,a_d\in\mathbb{C}$, or block-diagonal matrix in $\mathcal{M}_{md,md}(\mathbb{C})$ with blocks $a_1,\dots,a_d\in\mathcal{M}_m(\mathbb{C})$ along the main diagonal.

A^{T}, A^*	Transpose and Hermitian transpose of the matrix $A \in \mathcal{M}_{d,m}(\mathbb{C})$, respectively.
$\mathcal{C}(A,B)$	Controllability matrix of the pair $(A,B) \in \mathcal{M}_d(\mathbb{K}) \times \mathcal{M}_{d,m}(\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, given by $\mathcal{C}(A,B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{d-1}B \end{pmatrix} \in \mathcal{M}_{d,dm}(\mathbb{K})$.
$\rho(A)$	Spectral radius of the matrix $A \in \mathcal{M}_d(\mathbb{C})$.
$\det A$	Determinant of the matrix $A \in \mathcal{M}_d(\mathbb{C})$.
$\operatorname{Tr} A$	Trace of the matrix $A \in \mathcal{M}_d(\mathbb{C})$.
$\operatorname{Ran} A$	Range of the matrix $A \in \mathcal{M}_{d,m}(\mathbb{K})$, seen as a $\mathbb{K}$ vector space, for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.
$\operatorname{Ker} A$	Kernel of the matrix $A \in \mathcal{M}_{d,m}(\mathbb{K})$, seen as a $\mathbb{K}$ vector space, for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.
$\operatorname{rk} A$	Dimension of $\operatorname{Ran} A$.
$e_1,\dots,e_d$	Canonical basis of $\mathbb{R}^d$ or $\mathbb{C}^d$.
$\lvert\cdot\rvert_p$	ℓ_p norm of a vector in $\mathbb{R}^d$ or $\mathbb{C}^d$ and the induced matrix norm in $\mathcal{M}_{d,m}(\mathbb{R})$ or $\mathcal{M}_{d,m}(\mathbb{C})$. When p is omitted, it is assumed to be equal to 2.
$\prod_{j=1}^{N} A_j$	*Ordered* product $A_1A_2\cdots A_N$ of the matrices $A_1,\dots,A_N \in \mathcal{M}_d(\mathbb{C})$.
$\langle\cdot,\cdot\rangle_{\mathsf{H}}$	Inner product in the (real or complex) Hilbert space H. When H is complex, $\langle\cdot,\cdot\rangle_{\mathsf{H}}$ is assumed to be anti-linear in the first variable and linear in the second one. The index H is omitted when clear from the context.
$x \cdot y$	$\langle x,y\rangle_{\mathbb{R}^d}$ for $x,y \in \mathbb{R}^d$.
$\lVert\cdot\rVert_{\mathsf{X}}$	Norm in a (real or complex) Banach space X. The index X is omitted when clear from the context.
$x_n \rightharpoonup x$	The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x \in \mathsf{X}$ as $n \to +\infty$ in a Banach or Hilbert space X. Also used for weak-$\star$ convergence in the dual X'.
$D(T)$	Domain of the linear operator $T : D(T) \subset \mathsf{X} \to \mathsf{Y}$ from the Banach space X to the Banach space Y.
$\mathcal{L}(\mathsf{X},\mathsf{Y})$	Banach space of all bounded operators from the Banach space X to the Banach space Y, endowed with its usual induced norm.
$\mathcal{L}(\mathsf{X})$	$\mathcal{L}(\mathsf{X},\mathsf{X})$ for X a Banach space.
L^p	Usual Lebesgue space of p-integrable functions.
L^p_{loc}	Set of all locally p-integrable functions.
$W^{k,p}$	Usual Sobolev space of k-times weakly differentiable functions with derivatives in L^p, denoted by H^k when $p = 2$.
$\mathcal{C}(X,Y)$ or $\mathcal{C}^0(X,Y)$	Set of continuous functions from the topological space X to the topological space Y.
$\mathcal{C}^k(I,\mathsf{X})$	Set of k times differentiable X-valued functions defined on the interval $I \subset \mathbb{R}$, for $k \in \mathbb{N}$ and X a Banach space with its strong topology.
$\mathcal{C}^k_{\mathrm{c}}(I,\mathsf{X})$	Subset of $\mathcal{C}^k(I,\mathsf{X})$ of all the compactly supported functions.
$\mathcal{C}^k(I)$, $\mathcal{C}^k_{\mathrm{c}}(I)$	$\mathcal{C}^k(I,\mathbb{R})$ and $\mathcal{C}^k_{\mathrm{c}}(I,\mathbb{R})$, respectively.

Chapter 1

Introduction

1.1 Switched systems

In the past decades, several works have considered systems whose behavior is described by discrete and continuous variables in interaction, known as *hybrid systems* [12, 61, 75, 115, 121, 158]. Hybrid systems have attracted much research effort from different areas, such as engineering, computer science, and mathematics, due to both the interesting theoretical questions that their analysis arises and their numerous applications, for instance in process control, the automotive industry, power systems, air traffic control, or chemical processes. Typically, hybrid systems are a useful model whenever one considers a continuous physical process controlled by a discrete switching logic, usually implemented as an algorithm programmed in some embedded control device, which explains the increasing importance given to the study of hybrid systems recently.

A simple example of hybrid system is the temperature regulation system of a room [12, 121], which, in a simplified description, can be modeled in terms of two variables, the (continuous) temperature θ and the (discrete) state of the heater q, either “on” or “off”. The state of the heater determines the evolution of the temperature, and a switching logic controls when the heater is automatically switched on or off in terms of certain prescribed thresholds in the temperature. Another example, of greater practical interest, is a four-stroke gasoline automotive engine [22], in which the continuous physical variables representing power-train and air dynamics interact with a discrete variable describing in which of the four possible modes of operation the engine piston is.

In several applications, one is mostly interested in the behavior and properties of the continuous variable, neglecting the precise details of the dynamics driving the evolution of the discrete variable. In order to reflect such preference for the continuous variable in the mathematical model, one may regard the discrete variable only as modes or subsystems defining the evolution of the continuous variable and ignore its full dynamics by considering instead a certain class of switching patterns. These continuous systems with discrete switching events are known as *switched systems* [113,114,123,158,166]. Hence, in a switched system model, one is not interested in the time evolution of the discrete variable itself, but only on the effects of such evolution on the continuous variable.

Mathematically, a switched system in $\mathbb{R}^d$ can be described by a family of vector fields $f_k : \mathbb{R}^d \to \mathbb{R}^d$, $k \in \mathcal{J}$, where $\mathcal{J}$ is a set of indices (usually assumed to be finite), and by a piecewise constant function $\alpha : \mathbb{R}_+ \to \mathcal{J}$, as

$$\dot{x}(t) = f_{\alpha(t)}(x(t)), \qquad t \in \mathbb{R}_+. \tag{1.1}$$

The continuous state $x(t)$ is a vector in $\mathbb{R}^d$, or, more generally, belongs to some manifold M

or some Banach space X. The signal α is called *switching signal*. It is usually assumed to be piecewise constant (with finitely many discontinuities on any bounded interval), determining, for every time interval in which it is constant, which one among the vector fields f_k is driving the dynamics of the system. In general, α is not precisely known and one is interested instead in obtaining robust properties of the system (1.1) with respect to a certain class $\mathcal{G}$ of switching signals α. The mathematical model (1.1) can be modified to take into account state-dependent switching signals or possible discontinuities in the state variable x in some discrete set of times, among others (see, e.g., [113, 121]). One can also consider switched control systems, under the form

$$\dot{x}(t) = f_{\alpha(t)}(x(t), u(t)), \qquad t \in \mathbb{R}_+, \tag{1.2}$$

where $u(t) \in \mathbb{R}^m$ denotes a control input.

Switched systems have been studied in the literature from several different points of view, such as modeling [22], verification [50], controllability [174], observability [16], optimal control [30], stability, and stabilization [114]. In switched system models, the switching signal α can be assumed to be *controlled*, meaning that it can be imposed by the designer in order to achieve some prescribed goal, or *uncontrolled*, meaning that it is imposed by some external factor and cannot be modified by the designer. In the first case, one is usually interested in obtaining results guaranteeing the existence and characterizing switching signals that achieve a certain goal, such as controlling the system to a final state, designing a switching sequence in order to observe the state of the system, or stabilizing the system to the origin. In the second one, the aim is to obtain properties of the system that hold for all switching signals in a certain class, that may or may not contain constraints on the switching behavior.

The main and most interesting feature of switched systems (1.1) is that the interaction between the continuous dynamics and the switching signal may produce effects that are not present in the isolated continuous subsystems $\dot{x}(t) = f_k(x(t))$. The following example, which is classical in the literature of switched systems, shows that switching between exponentially stable linear subsystems may lead to unstable behavior.

Example 1.1. Consider the linear switched system

$$\dot{x}(t) = A_{\alpha(t)} x(t) \tag{1.3}$$

with $\alpha : \mathbb{R}_+ \to \{1, 2\}$ piecewise constant and $A_1, A_2 \in \mathcal{M}_2(\mathbb{R})$ given by

$$A_1 = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix}.$$

One immediately verifies that A_1 and A_2 are Hurwitz matrices, sharing the same eigenvalues $\lambda_1 = -1 + 3i$ and $\lambda_2 = -1 - 3i$, and hence both subsystems $\dot{x}(t) = A_1 x(t)$ and $\dot{x}(t) = A_2 x(t)$ are exponentially stable. Let $\alpha : \mathbb{R}_+ \to \{1, 2\}$ be the switching signal given by

$$\alpha(t) = \begin{cases} 1 & \text{if } t \in \left[k\frac{\pi}{3}, k\frac{\pi}{3} + \frac{\pi}{6}\right) \text{ for some } k \in \mathbb{N}, \\ 2 & \text{if } t \in \left[k\frac{\pi}{3} + \frac{\pi}{6}, (k+1)\frac{\pi}{3}\right) \text{ for some } k \in \mathbb{N}. \end{cases} \tag{1.4}$$

The solution of (1.3) with initial condition $x_0 = (0, 1)$ associated with such switching signal is given by

$$x(t) = \begin{cases} (-9)^k e^{-t} \begin{pmatrix} 3\sin(3t) \\ \cos(3t) \end{pmatrix}, & \text{if } t \in \left[k\frac{\pi}{3}, k\frac{\pi}{3} + \frac{\pi}{6}\right) \text{ for some } k \in \mathbb{N}, \\ 3(-9)^k e^{-t} \begin{pmatrix} \sin(3t) \\ -3\cos(3t) \end{pmatrix}, & \text{if } t \in \left[k\frac{\pi}{3} + \frac{\pi}{6}, (k+1)\frac{\pi}{3}\right) \text{ for some } k \in \mathbb{N}. \end{cases} \tag{1.5}$$

Since $9 > e^{\pi/3}$, one has $|x(t)| \to +\infty$ as $t \to +\infty$, and thus the switched system (1.3) is unstable. The trajectory (1.5) is represented in Figure 1.1.

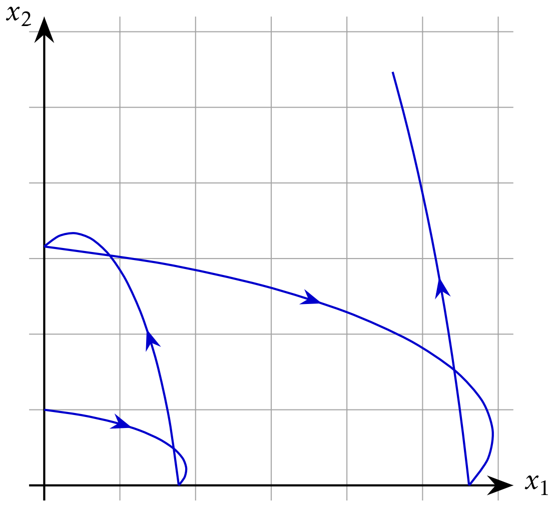


Figure 1.1: Trajectory of (1.3) with initial condition $x_0 = (0,1)$ associated with the switching signal α given by (1.4).

Example 1.1 can be easily modified to show that switching between unstable systems may lead to exponentially stable trajectories, and is only one among several examples depicting the fact that the dynamics of a switched system can differ much from those of its isolated subsystems.

Despite the major advances in the theory of switched systems, several important questions concerning their behavior remain open, even in the linear case. This is particularly true for switched systems with random switching signals and for infinite-dimensional switched systems, which have attracted much research effort recently [7, 17, 27, 28, 76, 79, 90, 111, 118, 149, 169]. This thesis presents, in Chapters 2, 3, and 4, new results on the stability of switched systems, both in infinite dimension with deterministic switching signals and in finite dimension with random switching signals, also considering the stabilization problem in the latter framework. We focus here on linear switched systems and assume that the switching signals are uncontrolled.

1.2 Persistently excited systems

An important class of switched systems, whose study was the main motivation for this thesis, is that of *persistently excited systems*. The introduction to such systems provided in this section is based on that of [46].

Consider a switched control system under the form (1.2) where the switching signal only affects the control input of the system by switching it on or off. This corresponds to the control system $\dot{x}(t) = f(x(t), \alpha(t)u(t))$, where $\alpha : \mathbb{R}_+ \to \{0,1\}$, or $\alpha : \mathbb{R}_+ \to [0,1]$ if one allows different levels of activation for the control input. When f is a linear map in (x,u), this system becomes

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \qquad t \in \mathbb{R}_+,$$

for some matrices $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$. If α takes its values in $\{0,1\}$, this is a switched system between the controlled dynamics $\dot{x} = Ax + Bu$ and the uncontrolled one $\dot{x} = Ax$.

The signal α may model several different phenomena, such as failure in the transmission from the controller to the plant, leading to instants of time at which the control is switched off; time-varying parameters affecting the control efficiency, leading to the effective application of a rescaled control $\alpha(t)u(t)$; allocation of control resources, activating the control only up to a certain fraction of its designed value, or only on certain time intervals; among other possible situations. Such issues are important from a practical point of view, in particular in systems controlled by wireless networks [93, 101, 102], where packet dropouts or communication constraints may degrade the control performance.

We assume here that the switching signal α is uncontrolled and that the only information one has on α is that it belongs to a certain class $\mathcal{G} \subset L^\infty(\mathbb{R}, [0,1])$. In order to have an interesting problem from the control point of view, the class $\mathcal{G}$ should be chosen in such a way that all signals $\alpha \in \mathcal{G}$ ensure a *sufficient amount of action* of the control u on the system. A condition normally used for this purpose (cf. e.g. [46, 49, 116, 135, 164]), which arises naturally in identification and adaptive control problems, is that of *persistence of excitation*, defined as follows.

Definition 1.2. Let T, μ be two positive constants with $T \geq \mu > 0$. A function $\alpha \in L^\infty(\mathbb{R}, [0,1])$ is said to be a (T,μ)*-persistently exciting signal* if, for every $t \in \mathbb{R}$, one has

$$\int_t^{t+T} \alpha(s) ds \geq \mu. \tag{1.6}$$

The set of all (T,μ)-persistently exciting signals is denoted by $\mathcal{G}(T,\mu)$. The family of linear control systems

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \qquad \alpha \in \mathcal{G}(T,\mu), \tag{1.7}$$

is called a *persistently excited system*.

Remark 1.3. Since the signal α in (1.7) is evaluated only on non-negative times, one may replace $\mathbb{R}$ by $\mathbb{R}_+$ in Definition 1.2 and consider $\mathcal{G}(T,\mu)$ as a class of signals defined on $\mathbb{R}_+$. However, it is convenient in some statements and proofs to consider persistently exciting signals α as defined on the whole real line in order to avoid cumbersome notations (this is the case, for instance, of Theorem 3.18 and Proposition 3.24).

The persistence of excitation condition (1.6) finds its origins in problems stemming from identification and adaptive control [9–11, 37]. In such situations, one is lead to consider the stability of linear systems of the kind $\dot{x}(t) = -P(t)x(t)$, $x(t) \in \mathbb{R}^d$, where the matrix $P(t)$ is symmetric positive semi-definite for every $t \in \mathbb{R}_+$. If P is also bounded and has a bounded derivative, a *necessary and sufficient* condition for the global exponential stability of $\dot{x}(t) = -P(t)x(t)$, proved in [135], is that P is also persistently exciting, in the sense that there exist $T \geq \mu > 0$ such that

$$\int_t^{t+T} \xi^{\mathrm{T}} P(s) \xi ds \geq \mu,$$

for all unitary vectors $\xi \in \mathbb{R}^d$ and all $t \geq 0$.

Still in the context of identification and adaptive control, the condition of persistence of excitation is useful when analyzing the convergence of certain identification methods for linear systems, where the identification error satisfies an equation of the form $\dot{x}(t) = -u(t)u(t)^{\mathrm{T}}x(t)$ [9, 11, 37, 162]. In this case, it can be shown that, under some regularity hypothesis on u, exponential stability of this system is equivalent to the existence of positive constants μ_1, μ_2, and T such that

$$\mu_1 \operatorname{Id}_d \leq \int_t^{t+T} u(s)u(s)^{\mathrm{T}} ds \leq \mu_2 \operatorname{Id}_d. \tag{1.8}$$

A question of practical importance in this case is to estimate the rate of exponential convergence to zero [9,37,162] and to compare different estimates [11]. It is also important to note that the right-hand side inequality in (1.8) is a necessary condition for the convergence to the origin of the trajectories of $\dot{x}(t) = -u(t)u(t)^{\mathrm{T}}x(t)$ [23].

Several interesting problems involve the study of some generalized form of the persistently excited system (1.7) [116]. One such problem is the control of spacecrafts with magnetic actuators [117], which can be described in a simplified form by the system

$$\dot{\omega}(t) = S(\omega(t))\omega(t) + g(t)u(t),$$

where $\omega(t) \in \mathbb{R}^3$ is the state variable, $u(t)$ is the control input, $S(\omega) \in \mathcal{M}_3(\mathbb{R})$ is a matrix depending linearly on $\omega \in \mathbb{R}^3$, and $g(t)$ is a time-varying matrix with $\operatorname{rk} g(t) < 3$ for all time t and satisfying some generalized persistent excitation condition. A feedback control for such system has been designed in [117] using persistence of excitation arguments. Further examples of systems similar to (1.7) where the persistent excitation condition appears are given in [116].

Before introducing the main problems of interest for the persistently excited system (1.7), let us recall some classical results concerning the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x \in \mathbb{R}^d,\ u \in \mathbb{R}^m \tag{1.9}$$

(cf. [36, 163]). Such system is said to be *controllable* in time $T > 0$ if, for every $x_0, x_1 \in \mathbb{R}^d$, there exists a control $u : [0, T] \to \mathbb{R}^m$ such that the unique solution of (1.9) with initial condition x_0 and control u satisfies $x(T) = x_1$, and, according to Kalman controllability criterion, this is equivalent to requiring that the controllability matrix

$$\mathcal{C}(A, B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{d-1}B \end{pmatrix} \in \mathcal{M}_{d,dm}(\mathbb{R})$$

has full rank. In this case, we also say that the pair of matrices $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ is controllable. We say that (1.9) is *stabilizable* by a linear feedback (or that the pair of matrices (A, B) is stabilizable) if there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the closed-loop system $\dot{x}(t) = (A+BK)x(t)$ is asymptotically stable, which is equivalent to requiring the matrix $A+BK$ to be Hurwitz. This is the case if and only if, up to a linear change of variables, A and B can be written under the form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

with A_3 Hurwitz and (A_1, B_1) controllable. We also recall the following result, which is an immediate consequence of the Pole-shifting Theorem and Kalman decomposition (see, e.g., [163, Lemma 3.3.3 and Theorem 13]).

Proposition 1.4. *Let $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$. The following assertions are equivalent.*

(a) *(A, B) is controllable.*

(b) *For every monic polynomial χ of degree d, there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that χ is the characteristic polynomial of $A + BK$.*

(c) *For every $\gamma > 0$, there exist $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $C > 0$ such that every solution x of the closed-loop system $\dot{x}(t) = (A + BK)x(t)$ satisfies*

$$|x(t)| \leq Ce^{-\gamma t}|x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

(d) *For every $\gamma > 0$, there exist $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $C > 0$ such that every solution x of the closed-loop system $\dot{x}(t) = (A + BK)x(t)$ satisfies*

$$|x(t)| \geq Ce^{\gamma t}|x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

1.2.1 Finite-dimensional persistently excited systems

We review in this section several results for the persistently excited system (1.7) in finite dimension. Notice first that, thanks to Carathéodory's Theorem (see, e.g., [83, Section I.5]), for every $T,\mu > 0$ satisfying $T \geq \mu$, $\alpha \in \mathcal{G}(T,\mu)$ (or, more generally, for every $\alpha \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$), $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^m)$, and $x_0 \in \mathbb{R}^d$, (1.7) admits a unique absolutely continuous solution $x : \mathbb{R}_+ \to \mathbb{R}^d$ with initial condition $x(0) = x_0$. Carathéodory's Theorem can also be applied to the closed-loop system $\dot{x}(t) = (A + \alpha(t)BK)x(t)$, obtained by choosing $u(t) = Kx(t)$ for some $K \in \mathcal{M}_{m,d}(\mathbb{R})$, yielding existence and uniqueness of its solutions.

1.2.1.1 The controllability problem

The first problem we consider is the controllability of (1.7), defined as follows.

Definition 1.5. Let $\tau,T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. We say that (1.7) is *controllable* in time τ if, for every $\alpha \in \mathcal{G}(T,\mu)$ and $x_0,x_1 \in \mathbb{R}^d$, there exists $u \in L^1((0,\tau),\mathbb{R}^m)$ such that the unique solution x of (1.7) with initial condition $x(0) = x_0$ and control u satisfies $x(\tau) = x_1$.

This is a simple formulation of the controllability problem for (1.7), where one assumes having full knownledge of the signal α. A necessary condition for the controllability of (1.7) in some time $\tau > 0$ is the controllability of the pair $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$, since the persistently exciting signal constantly equal to 1 is in the class $\mathcal{G}(T,\mu)$ for every $T,\mu \in \mathbb{R}_+^*$ with $T \geq \mu$. Moreover, since there exist signals $\alpha \in \mathcal{G}(T,\mu)$ that are identically zero on $(0,T-\mu)$, another necessary condition for the controllability of (1.7) in time τ is that $\tau > T-\mu$. These conditions turn out to be sufficient as well, as shown by the following result from [39], whose proof is very similar to the classical proof of the Kalman controllability criterion.

Proposition 1.6 [39, Proposition 4]. *Let $\tau,T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$ and $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$. Then (1.7) is controllable in time τ if and only if the pair (A,B) is controllable and $\tau > T - \mu$.*

1.2.1.2 The stabilizability problem

A much more interesting and less trivial problem is that of the *uniform* stabilization of (1.7) by a linear feedback law, which consists on finding $K \in \mathcal{M}_{m,d}(\mathbb{R})$, depending only on A, B, T, and μ, such that the closed-loop system

$$\dot{x}(t) = (A + \alpha(t)BK)x(t) \tag{1.10}$$

is globally exponentially stable for every $\alpha \in \mathcal{G}(T,\mu)$.

Definition 1.7. Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, and $T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. We say that $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T,μ)-*stabilizer* for (1.7) if there exist constants $C,\gamma > 0$ such that, for every $x_0 \in \mathbb{R}^d$ and $\alpha \in \mathcal{G}(T,\mu)$, the unique solution x of (1.10) satisfies

$$\lvert x(t)\rvert \leq Ce^{-\gamma t}\lvert x_0\rvert, \qquad \forall t \in \mathbb{R}_+.$$

Remark 1.8. Thanks to Fenichel's Uniformity Lemma (see, e.g., [52, Lemma 5.2.7]), $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T,μ)-stabilizer for (1.7) if and only if, for every $x_0 \in \mathbb{R}^d$ and $\alpha \in \mathcal{G}(T,\mu)$, the unique solution x of (1.10) satisfies $\limsup_{t\to+\infty} \lvert x(t)\rvert = 0$.

The uniform stabilizability of (1.7) by linear feedback laws has been addressed in several works in the literature [38, 39, 45, 49, 126, 128]. We review here the most important results on this problem, which served as motivation and starting point for this thesis.

Notice that, if there exists a (T,μ)-stabilizer K for (1.7), then in particular the linear time-invariant system $\dot{x}(t) = (A+BK)x(t)$ is asymptotically stable, which means that the stabilizability of the pair (A,B) is a necessary condition for the existence of a (T,μ)-stabilizer for (1.7). The first result providing a sufficient condition for the existence of a (T,μ)-stabilizer is the following, proved in [38, 39] (see also [10]). Recall that a matrix $A \in \mathcal{M}_d(\mathbb{R})$ is said to be *neutrally stable* if its eigenvalues have non-positive real part and those with real part zero have trivial corresponding Jordan blocks, which is equivalent to the stability (possibly non-asymptotic) of the linear system $\dot{x}(t) = Ax(t)$.

Theorem 1.9 [39, Theorem 7]**.** *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$. Suppose that the pair (A,B) is stabilizable and that the matrix A is neutrally stable. Then there exists a matrix $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for every $T,\mu \in \mathbb{R}_+^*$ with $T \geq \mu$, K is a (T,μ)-stabilizer for (1.7).*

The first step of the proof of Theorem 1.9 is to reduce to the case where (A,B) is controllable and A is skew-symmetric, which is possible since one only has to stabilize the system on the sum of all the eigenspaces of A associated with eigenvalues of real part zero, in which the restriction of A can be put under a skew-symmetric form by a linear change of variables. Thus, Theorem 1.9 follows from the following result.

Proposition 1.10. *Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, and $T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Suppose that the pair (A,B) is controllable and that the matrix A is skew-symmetric. Then $K = -B^{\mathrm{T}} \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T,μ)-stabilizer for (1.7).*

The choice of K in Proposition 1.10 leads to the system $\dot{x} = (A - \alpha(t)BB^{\mathrm{T}})x$, for which one may prove that $V(x) = \lvert x\rvert^2$ is a weak Lyapunov function. One computes $\frac{d}{dt}V(x(t)) = -2\alpha(t)\left\lvert B^{\mathrm{T}}x(t)\right\rvert^2$ and uses a Lasalle-type argument to conclude; for the details of the proof, we refer to [39].

An interesting feature of Theorem 1.9 is that the feedback matrix K does not depend on T or μ, which comes from the fact that $K = -B^{\mathrm{T}}$ stabilizes (1.7) for every $T,\mu \in \mathbb{R}_+^*$ with $T \geq \mu$ under the hypotheses of Proposition 1.10. However, Theorem 1.9 deals only with with control systems whose uncontrolled dynamics $\dot{x} = Ax$ are already stable, and it is obviously also interesting to consider the stabilizability of systems whose uncontrolled dynamics are not necessarily stable. This has been done in [49], where the following improvement of Theorem 1.9 has been proved.

Theorem 1.11 [49, Theorem 3.2]**.** *Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, and $T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Suppose that the pair (A,B) is stabilizable and that the eigenvalues of A have non-positive real part. Then there exists a (T,μ)-stabilizer for (1.7).*

Theorem 1.11 has been proved for controllable pairs (A,B) and in the single-input case $m = 1$ in [49, Theorem 3.2]. The multi-input case follows by an induction on the number of inputs [46, Theorem 2.9], and the fact that one can reduce the case of stabilizable pairs (A,B) to that of controllable pairs follows, e.g., from [126, Lemma B.1]. It improves Theorem 1.9 in the sense that A is no longer assumed to be neutrally stable, and hence trajectories of the uncontrolled system $\dot{x}(t) = Ax(t)$ may diverge, even though such divergence can only be polynomial in time. However, the feedback matrix K depends in general on T and μ.

The proof of Theorem 1.11 provided in [49] relies on a time-contraction procedure and on the compactness of $L^\infty(\mathbb{R},[0,1])$ with respect to the weak-$\star$ topology of $L^\infty(\mathbb{R},\mathbb{R})$. The time-contraction procedure transforms the integral condition of persistence of excitation (1.6) into a pointwise one in the limit as the time-contraction parameter tends to $+\infty$. One can define a limit system, which can be shown to be stable via a suitable Lyapunov function, and an approximation result allows one to conclude the stability of a time-contracted system from the stability of the limit system.

The time-contraction technique used in [49] is also well-adapted to take into account delays in the feedback loop, since such delays are reduced by the time-contraction procedure and vanish in the limit as the time-contraction parameter tends to $+\infty$. More precisely, consider (1.7) and assume that, instead of applying an instantaneous feedback $u(t) = Kx(t)$, one applies a delayed feedback $u(t) = Kx(t - \tau(t))$, where $\tau \in L^\infty(\mathbb{R}, \mathcal{T})$ for some bounded set $\mathcal{T} \subset \mathbb{R}_+$. The closed-loop system (1.10) becomes

$$\dot{x}(t) = Ax(t) + \alpha(t)BKx(t - \tau(t)), \qquad \alpha \in \mathcal{G}(T, \mu),\ \tau \in L^\infty(\mathbb{R}, \mathcal{T}). \tag{1.11}$$

Thanks to Carathéodory's Theorem for delayed equations (see, e.g., [86, Section 2.6 and Chapter 6, Theorem 1.1]), for every $T, \mu \in \mathbb{R}_+^*$ with $T \geq \mu$, $\alpha \in \mathcal{G}(T, \mu)$, $\tau \in L^\infty(\mathbb{R}, \mathcal{T})$, and $x_0 \in \mathcal{C}^0([-r, 0], \mathbb{R}^d)$, where $r = \sup \mathcal{T}$, (1.11) admits a unique continuous solution x defined on $[-r, +\infty)$, which is absolutely continuous on $\mathbb{R}_+$, coincides with x_0 on $[-r, 0]$, and satisfies (1.11) for almost every $t \in \mathbb{R}_+$. One can then extend the definition of (T, μ)-stabilizer to (1.11) as follows.

Definition 1.12. Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$, $\mathcal{T} \subset \mathbb{R}_+$ be bounded, and $r = \sup \mathcal{T}$. We say that $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a $(T, \mu, \mathcal{T})$-*stabilizer* for (1.11) if there exist constants $C, \gamma > 0$ such that, for every $x_0 \in \mathcal{C}^0([-r, 0], \mathbb{R}^d)$, $\alpha \in \mathcal{G}(T, \mu)$, and $\tau \in L^\infty(\mathbb{R}, \mathcal{T})$, the unique solution x of (1.11) satisfies

$$\left|x(t)\right| \leq Ce^{-\gamma t} \sup_{s \in [-r, 0]} \left|x_0(s)\right|, \qquad \forall t \in \mathbb{R}_+.$$

Notice that the dynamics of (1.11) is infinite-dimensional, taking place in the Banach space $\mathcal{C}^0([-r, 0], \mathbb{R}^d)$, and hence, differently from Remark 1.8, Fenichel's Uniformity Lemma cannot be applied here.

The following generalization of Theorem 1.11 holds.

Theorem 1.13 [126, Theorem 2.5]**.** *Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, and $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Suppose that the pair (A, B) is stabilizable and that the eigenvalues of A have non-positive real part. Then, for every $\tau_0 \geq 0$, there exists a neighborhood $\mathcal{T}$ of τ_0 in $\mathbb{R}_+$ and a $(T, \mu, \mathcal{T})$-stabilizer for* (1.11)*.*

In order to highlight the time-contraction argument used in the proofs of Theorems 1.11 and 1.13, we provide the proof of the latter in the particular case of the d-integrator, corresponding to $A = J_d$ and $B = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}}$, where J_d denotes the $d \times d$ Jordan block

$$J_d = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{1.12}$$

(see [126, Theorem 3.1]). This particular case is interesting since it contains most of the difficulties of the general case. Furthermore, we can give in this case a stronger result, showing the existence of a $(T, \mu, \mathcal{T})$-stabilizer for *any* bounded interval $\mathcal{T} \subset \mathbb{R}_+$, and not only for perturbations around a certain value as in the general case of Theorem 1.13.

Proposition 1.14 [126, Theorem 3.1]. *Let $A = J_d$, $B = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^{\mathrm{T}} \in \mathcal{M}_{d,1}(\mathbb{R})$, $r \geq 0$, and $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Then there exists a $(T, \mu, [0, r])$-stabilizer $K \in \mathcal{M}_{1,d}(\mathbb{R})$ for (1.11).*

In order to prove Proposition 1.14, we need the following continuity result for solutions of delayed equations.

Lemma 1.15 [126, Lemma A.1]. *Let $A \in \mathcal{M}_d(\mathbb{R})$, $B \in L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{R}))$, $r \geq 0$, and $\tau \in L^\infty(\mathbb{R}, [0, r])$. Consider the system*

$$\dot{x}(t) = Ax(t) + B(t)x(t - \tau(t)). \tag{1.13}$$

Denote by $x(\cdot; \tau, x_0, B)$ its solution with initial condition $x_0 \in \mathcal{C}^0([-r, 0], \mathbb{R}^d)$. Let $(\tau_n)_{n\in\mathbb{N}^*}$ be a sequence on $L^\infty(\mathbb{R}, [0, r])$ such that $\tau_n(t) \to 0$ as $n \to +\infty$ uniformly on $\mathbb{R}$. Suppose that $(x_0^{(n)})_{n\in\mathbb{N}^*}$ is a sequence of functions in $\mathcal{C}^0([-r, 0], \mathbb{R}^d)$ and $(B_n)_{n\in\mathbb{N}^*}$ is a bounded sequence on $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{R}))$ satisfying

(a) $\lim_{n\to+\infty} x_0^{(n)}(0) = x_0^\star$ *for some* $x_0^\star \in \mathbb{R}^d$;

(b) *there exists* $\Lambda > 0$ *such that* $\left|x_0^{(n)}(t)\right| \leq \Lambda$ *for all* $n \in \mathbb{N}^*$ *and all* $t \in [-r, 0]$;

(c) $B_n \rightharpoonup B_\star$ *weakly-$\star$ as* $n \to +\infty$, *for some* $B_\star \in L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{R}))$.

Then $x(t; \tau_n, x_0^{(n)}, B_n) \to x(t; 0, x_0^\star, B_\star)$ as $n \to +\infty$, uniformly on compact time intervals in $\mathbb{R}_+$.

The proof we provide below for Proposition 1.14 is that of [126].

Proof of Proposition 1.14. The proof follows the same idea of that of [49, Theorem 3.1]: we first perform a change of variables corresponding to a time contraction in order to relate $(T, \mu, [0, r])$-stabilizers to $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizers for $\nu > 0$. We then study the stabilizability of a certain limit system, and this allows us to conclude the stabilizability of the original system for a certain $\nu > 0$ large enough, thanks to Lemma 1.15. In this proof, the unique solution x of (1.11) associated with the delay $\tau \in L^\infty(\mathbb{R}, [0, r])$, the initial condition $x_0 \in \mathcal{C}^0([-r, 0], \mathbb{R}^d)$, the persistently exciting signal $\alpha \in \mathcal{G}(T, \mu)$, and the feedback matrix $K \in \mathcal{M}_{1,d}(\mathbb{R})$, is denoted by $x(\cdot; \tau, x_0, \alpha, K)$.

Step 1. *Time contraction.*
For $\nu > 0$, define $D_{d,\nu} = \operatorname{diag}(\nu^{d-1}, \ldots, \nu, 1) \in \mathcal{M}_d(\mathbb{R})$, which satisfies the relations $\nu D_{d,\nu}^{-1} J_d D_{d,\nu} = J_d$ and $D_{d,\nu} B = B$. Noting, for simplicity, $x(t) = x(t; \tau, x_0, \alpha, K)$, and defining $x_\nu(t) = D_{d,\nu}^{-1} x(\nu t)$, one obtains that x_ν satisfies

$$\frac{d}{dt} x_\nu(t) = J_d x_\nu(t) + \alpha(\nu t) \nu B K D_{d,\nu} x_\nu\left(t - \frac{\tau(\nu t)}{\nu}\right) \tag{1.14}$$

and hence

$$x_\nu(t) = x\left(t; \frac{\tau(\nu \cdot)}{\nu}, D_{d,\nu}^{-1} x_0(\nu \cdot), \alpha_\nu, \nu K D_{d,\nu}\right)$$

with $\alpha_\nu(t) = \alpha(\nu t)$, which is a $(T/\nu, \mu/\nu)$-signal. Thus K is a $(T, \mu, [0, r])$-stabilizer for (1.11) if and only if $\nu K D_{d,\nu}$ is a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer. This equivalence is crucial in the sequel: instead of looking for a $(T, \mu, [0, r])$-stabilizer for (1.11), we look for a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for a certain $\nu > 0$ large enough. The technique is thus to study a certain limit system obtained as $\nu \to +\infty$, find a stabilizer for such non-delayed system, and finally show that such stabilizer is a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for (1.11) if ν is large enough.

Step 2. *Limit system.*
Consider the system

$$\dot{x}(t) = J_d x(t) + \alpha_\star(t) BKx(t), \qquad \alpha_\star \in L^\infty(\mathbb{R}, [\mu/T, 1]). \tag{1.15}$$

It has been proved in [49, Theorem 3.1], using a result from [71] attributed to W. Dayawansa (see also [72, Lemma 2.1]), that one can find $K \in \mathcal{M}_{1,d}(\mathbb{R})$ and a positive definite matrix $S \in \mathcal{M}_d(\mathbb{R})$, both independent of the particular signal $\alpha_\star \in L^\infty(\mathbb{R}, [\mu/T, 1])$, such that (1.15) is globally uniformly exponentially stable and $V(x) = x^{\mathrm{T}} Sx$ decreases along all trajectories of (1.15), uniformly with respect to $\alpha_\star$. In particular, there exists a time σ such that every trajectory of (1.15) starting in $B_2^V = \{x \in \mathbb{R}^d \mid V(x) \leq 2\}$ at time 0 lies in $B_1^V = \{x \in \mathbb{R}^d \mid V(x) \leq 1\}$ for every time larger than σ.

Step 3. *Study of* (1.14) *through the limit system.*
We wish to deduce from the conclusion obtained in the previous step that (1.11) admits a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer for some $\nu > 0$ large enough. We claim that, for some $\nu > 0$ large enough, every trajectory of

$$\dot{x}(t) = J_d x(t) + \alpha(t) BKx(t - \tau(t)), \qquad \alpha \in \mathcal{G}(T/\nu, \mu/\nu),\ \tau \in L^\infty(\mathbb{R}, [0, r/\nu]),$$

with initial condition $x_0 \in \mathcal{C}^0([-r/\nu, 0], B_2^V)$ stays in B_1^V for every time larger than 2σ. In particular, by homogeneity, this will imply that K is a $(T/\nu, \mu/\nu, [0, r/\nu])$-stabilizer of (1.11) and thus $\nu^{-1} K D_{d,\nu}^{-1}$ is a $(T, \mu, [0, r])$-stabilizer, concluding the proof. To prove this, assume, by contradiction, that for every $n \in \mathbb{N}^*$ there exist $\tau_n \in L^\infty(\mathbb{R}, [0, r/n])$, $x_0^{(n)} \in \mathcal{C}^0([-r/n, 0], B_2^V)$, $\alpha_n \in \mathcal{G}(T/n, \mu/n)$, and $t_n \in [2\sigma, 4\sigma]$ such that

$$x\left(t_n; \tau_n, x_0^{(n)}, \alpha_n, K\right) \notin B_1^V. \tag{1.16}$$

Up to the extraction of a subsequence, we can suppose that, as $n \to +\infty$, $t_n \to t_\star \in [2\sigma, 4\sigma]$, $x_0^{(n)}(0) \to x_0^\star \in B_2^V$, and $\alpha_n \rightharpoonup \alpha_\star \in L^\infty(\mathbb{R}, [0, 1])$ weakly-$\star$; we also note that $\tau_n(t) \to 0$ as $n \to +\infty$ uniformly on $t \in \mathbb{R}_+$. Then, by Lemma 1.15, we obtain that $x\left(t_n; \tau_n, x_0^{(n)}, \alpha_n, K\right)$ converges to $x(t_\star; 0, x_0^\star, \alpha_\star, K)$ as $n \to +\infty$. We also note that, by [49, Lemma 2.5], $\alpha_\star(t) \geq \mu/T$ almost everywhere in $\mathbb{R}$, and so, by our previous study of (1.15), since $t_\star \geq 2\sigma$, by homogeneity, we have

$$V(x(t_\star; 0, x_0^\star, \alpha_\star, K)) \leq \frac{1}{2}.$$

This contradicts (1.16), establishing the desired result. ■

Let us now provide a sketch of the proof of Theorem 1.13. It suffices to consider the case where (A, B) is controllable, $m = 1$, and all the eigenvalues of A have real part zero [126, Appendix B]. In such situation, up to a linear change of coordinates transforming A into its real Jordan canonical form, (1.11) becomes

$$\begin{cases} \dot{x}_0(t) = J_{r_0} x_0(t) + \alpha(t) b^0 Kx(t - \tau(t)), & x_0(t) \in \mathbb{R}^{r_0}, \\ \dot{x}_j(t) = (\omega_j A^{(j)} + J_{r_j}^{\mathbb{C}}) x_j(t) + \alpha(t) b^j Kx(t - \tau(t)), & x_j(t) \in \mathbb{R}^{2r_j},\ j \in [\![1, h]\!], \end{cases}$$

where the spectrum of A is $\sigma(A) = \{\pm i\omega_j,\ j = j_0, j_0 + 1, \ldots, h\}$ with all the $\omega_j \geq 0$ distinct, $j_0 = 1$ if $0 \notin \sigma(A)$, $j_0 = 0$ and $\omega_0 = 0$ otherwise; r_j is the algebraic multiplicity of the eigenvalue

$i\omega_j$ (with $r_0 = 0$ if $0 \notin \sigma(A)$); J_{r_0} is the real Jordan block defined in (1.12); $J_n^{\mathbb{C}} \in \mathcal{M}_{2n}(\mathbb{R})$ is the Jordan block for complex eigenvalues,

$$J_n^{\mathbb{C}} = \begin{pmatrix} 0_{2\times 2} & \mathrm{Id}_2 & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & \mathrm{Id}_2 & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \mathrm{Id}_2 & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & \mathrm{Id}_2 \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \end{pmatrix},$$

i.e., $J_n^{\mathbb{C}} = J_n \otimes \mathrm{Id}_2$ in terms of the Kronecker product; $A^{(j)} = \mathrm{diag}(A_0, \dots, A_0) \in \mathcal{M}_{2r_j}(\mathbb{R})$ with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

and b^0 and b^j are respectively the vectors of $\mathbb{R}^{r_0}$ and $\mathbb{R}^{2r_j}$ with all the coordinates equal to zero except the last one that is equal to one. The idea now follows the case of the d-integrator considered in Proposition 1.14. We write $K = \begin{pmatrix} K_0 & K_1 & \dots & K_h \end{pmatrix}$ with $K_0 \in \mathcal{M}_{1,r_0}(\mathbb{R})$, $K_j \in \mathcal{M}_{1,2r_j}(\mathbb{R})$, $j \in [\![1, h]\!]$, and perform the change of variables given by

$$\begin{aligned} y_0(t) &= D_{r_0,\nu}^{-1} x_0(\nu t), \\ y_j(t) &= (D_{r_j,\nu}^{\mathbb{C}})^{-1} e^{-\nu t \omega_j A^{(j)}} x_j(\nu t), \qquad j \in [\![1, h]\!], \end{aligned}$$

with $D_{r_j,\nu}^{\mathbb{C}} = D_{r_j,\nu} \otimes \mathrm{Id}_2$. The system satisfied by the new variables $y_0, \dots, y_h$ is

$$\begin{cases} \dot{y}_0(t) = J_{r_0} y_0(t) + \alpha_\nu(t) b^0 \left[K_{0,\nu} y_0\left(t - \frac{\tau(\nu t)}{\nu}\right) + \sum_{\ell=1}^{h} K_{\ell,\nu} e^{(\nu t - \tau(\nu t))\omega_\ell A^{(\ell)}} y_\ell\left(t - \frac{\tau(\nu t)}{\nu}\right)\right], \\ \dot{y}_j(t) = J_{r_j}^{\mathbb{C}} y_j(t) + \alpha_\nu(t) e^{-\nu t \omega_j A^{(j)}} b^j \left[K_{0,\nu} y_0\left(t - \frac{\tau(\nu t)}{\nu}\right) + \sum_{\ell=1}^{h} K_{\ell,\nu} e^{(\nu t - \tau(\nu t))\omega_\ell A^{(\ell)}} y_\ell\left(t - \frac{\tau(\nu t)}{\nu}\right)\right], \\ \hfill j \in [\![1, h]\!], \end{cases} \tag{1.17}$$

with $\alpha_\nu(t) = \alpha(\nu t)$, $K_{0,\nu} = \nu K_0 D_{r_0,\nu}$, and $K_{\ell,\nu} = \nu K_\ell D_{r_\ell,\nu}^{\mathbb{C}}$ for $\ell \in [\![1, h]\!]$. As in the case of the d-integrator, $K = \begin{pmatrix} K_0 & K_1 & \cdots & K_h \end{pmatrix}$ is a $(T, \mu, \mathcal{T})$-stabilizer for (1.11) if and only if $K_\nu = \begin{pmatrix} K_{0,\nu} & K_{1,\nu} & \cdots & K_{h,\nu} \end{pmatrix}$ is a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for (1.17), where $\mathcal{T}/\nu = \{t/\nu \mid t \in \mathcal{T}\}$.

We look for a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer of (1.17) under the form $K_\nu = \begin{pmatrix} K_{0,\nu} & \cdots & K_{h,\nu} \end{pmatrix}$ with

$$K_{j,\nu} = \mathcal{K}_j \otimes b_0^{\mathrm{T}} e^{\tau_0 \omega_j A_0}, \qquad \mathcal{K}_j \in \mathcal{M}_{1,r_j}(\mathbb{R})$$

for $j \in [\![1, h]\!]$ and $K_{0,\nu} = \mathcal{K}_0$, where $b_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$. We have that $K_{j,\nu} e^{(\nu t - \tau(\nu t))\omega_j A^{(j)}} = \mathcal{K}_j \otimes b_0^{\mathrm{T}} e^{\nu t \omega_j A_0} + \mathcal{K}_j \otimes \left[b_0^{\mathrm{T}} e^{\nu t \omega_j A_0} \left(e^{-(\tau(\nu t) - \tau_0)\omega_j A_0} - \mathrm{Id}_2 \right)\right]$. Denoting by $\tilde{b}^j \in \mathbb{R}^{r_j}$ the vector with all coordinates equal to zero except the last one that is equal to one, we have $b^j = \tilde{b}^j \otimes b_0$, and thus $e^{-\nu t\omega_j A^{(j)}} b^j = \tilde{b}_j \otimes e^{-\nu t \omega_j A_0} b_0$. We finally write, for $j, \ell \in [\![1, h]\!]$,

$$\begin{aligned}
C_{00}^{(\nu)}(t) &= \alpha_\nu(t), \\
C_{0j}^{(\nu)}(t) &= \alpha_\nu(t) b_0^{\mathrm{T}} e^{\nu t \omega_j A_0}, \\
C_{j0}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0, \\
C_{j\ell}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0 b_0^{\mathrm{T}} e^{\nu t \omega_j A_0}, \\
P_{00}^{(\nu)}(t) &= P_{j0}^{(\nu)}(t) = 0, \\
P_{0j}^{(\nu)}(t) &= \alpha_\nu(t) b_0^{\mathrm{T}} e^{\nu t \omega_j A_0} \left[e^{-(\tau(\nu t) - \tau_0)\omega_j A_0} - \mathrm{Id}_2\right], \\
P_{j\ell}^{(\nu)}(t) &= \alpha_\nu(t) e^{-\nu t \omega_j A_0} b_0 b_0^{\mathrm{T}} e^{\nu t \omega_\ell A_0} \left[e^{-(\tau(\nu t) - \tau_0)\omega_\ell A_0} - \mathrm{Id}_2\right],
\end{aligned}$$

and thus (1.17) can be written under the form

$$\begin{cases}
\dot{y}_0(t) = J_{r_0} y_0(t) + \displaystyle\sum_{\ell=0}^{h} \left[b^0 \mathcal{K}_\ell \otimes \left(C_{0\ell}^{(\nu)}(t) + P_{0\ell}^{(\nu)}(t)\right)\right] y_\ell\left(t - \frac{\tau(\nu t)}{\nu}\right), \\
\dot{y}_j(t) = J_{r_j}^{\mathrm{C}} y_j(t) + \displaystyle\sum_{\ell=0}^{h} \left[\tilde{b}^j \mathcal{K}_\ell \otimes \left(C_{j\ell}^{(\nu)}(t) + P_{j\ell}^{(\nu)}(t)\right)\right] y_\ell\left(t - \frac{\tau(\nu t)}{\nu}\right), \qquad j \in [\![1, h]\!].
\end{cases} \tag{1.18}$$

We can arrange all the matrices $C_{j\ell}^{(\nu)}$ in a $(2h+1-j_0) \times (2h+1-j_0)$ symmetric matrix and all the matrices $P_{j\ell}^{(\nu)}$ in a $(2h+1-j_0) \times (2h+1-j_0)$ matrix respectively as

$$C^{(\nu)}(t) = \left(C_{j\ell}^{(\nu)}(t)\right)_{j_0 \leq j, \ell \leq h}, \qquad P^{(\nu)}(t) = \left(P_{j\ell}^{(\nu)}(t)\right)_{j_0 \leq j, \ell \leq h}.$$

We are now in a situation similar to the case of the d-integrator, where the scalar switching signal α is replaced by the matrix $C^{(\nu)} + P^{(\nu)}$. Taking $\mathcal{T}$ under the form $\mathcal{T} = [\tau_0 - r, \tau_0 + r] \cap \mathbb{R}_+$ for a certain $r > 0$ to be chosen, one has

$$\left|P_{j\ell}^{(\nu)}(t)\right| \leq \left|e^{-(\tau(\nu t) - \tau_0)\omega_j A_0} - \mathrm{Id}_2\right| = \sqrt{2\left[1 - \cos\left((\tau(\nu t) - \tau_0)\omega_j\right)\right]} \leq \left|(\tau(\nu t) - \tau_0)\omega_j\right| \leq r\Omega$$

with $\Omega = \max\{\omega_j \mid j = j_0, \dots, h\}$, and, in particular, if $r > 0$ is small, $P^{(\nu)}$ is also small. As in the proof of Proposition 1.14, we can define a limit system for (1.18) as $\nu \to +\infty$, which is stabilizable by a similar argument if $r > 0$ is small enough, and Lemma 1.15 allows one to conclude in the same manner as for the d-integrator that (1.17) admits a $(T/\nu, \mu/\nu, \mathcal{T}/\nu)$-stabilizer for some $\nu > 0$ large enough and some $r > 0$ small enough.

Theorems 1.9 and 1.11 require more assumptions than the stabilizability of the pair (A, B) in order to conclude the existence of a (T, μ)-stabilizer for (1.7). However, since the class $\mathcal{G}(T, T)$ contains only the function equal to 1 almost everywhere on $\mathbb{R}$, such extra assumptions are not necessary for the existence of (T, T)-stabilizers for (1.7), and a natural question is whether one really needs any extra assumption at all in order to obtain the existence of (T, μ)-stabilizers for (1.7). Such question has been addressed in [49] in the single-input case and the answer turns out to depend on the ratio μ/T.

Proposition 1.16 [49, Propositions 4.4 and 4.5]**.**

(a) *Let $d \in \mathbb{N}^*$. There exists $\rho^\star \in (0, 1)$ depending only on d such that, for every $T, \mu \in \mathbb{R}_+^*$ with $T \geq \mu$ and $\mu/T > \rho^\star$ and $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,1}(\mathbb{R})$ controllable, (1.7) admits a (T, μ)-stabilizer.*

(b) *There exists* $\rho_\star \in (0,1)$ *such that, for every* $T, \mu \in \mathbb{R}_+^*$ *with* $\mu/T < \rho_\star$ *and* $(A,B) \in \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_{2,1}(\mathbb{R})$ *controllable, if* $\lambda > 0$ *is large enough, then the system* $\dot{x}(t) = (A + \lambda \operatorname{Id}_2)x(t) + \alpha(t)Bu(t)$ *does not admit a* (T,μ)*-stabilizer.*

The main idea for the proof of Proposition 1.16(a) is that, if K asymptotically stabilizes the closed-loop system $\dot{x}(t) = (A+BK)x(t)$, a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ for such system can still be used as a Lyapunov function for (1.10) if μ/T is large enough, and the uniformity of $\rho^\star$ follows by taking (A,B) under some normal form. On the other hand, Proposition 1.16(b) is proved by explicitly constructing, for each $K \in \mathcal{M}_{1,2}(\mathbb{R})$, a periodic persistently exciting signal α taking values in $\{0,1\}$ that destabilizes the closed-loop system $\dot{x}(t) = (A + \lambda \operatorname{Id}_2 + \alpha(t)BK)x(t)$, where λ is chosen only in terms of A and B.

Let us point out that another interesting and related stabilization problem is to find out whether (1.7) can be stabilized by feedback laws more general than the linear feedback $u(t) = Kx(t)$ for a constant matrix K. Such problem has been considered in several works in the literature [151,164,165,173], where one generally observes or estimates the signal α and constructs a time-varying feedback law depending on α or its estimation. Such stabilization results usually require more assumptions on the signal α, such as $\mathcal{C}^k$ regularity for some $k \in \mathbb{N}$ and boundedness of some of its derivatives, which are not needed in the above results. As an example, we provide here the stabilization result from [164].

Consider the single-input control system

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t) \tag{1.19}$$

with $x(t) \in \mathbb{R}^d$, $u(t) \in \mathbb{R}$, $A \in \mathcal{M}_d(\mathbb{R})$, and $B \in \mathcal{M}_{d,1}(\mathbb{R})$. Assume that $\alpha : \mathbb{R} \to \mathbb{R}$ is measurable and that there exist $T, \mu_1, \mu_2 > 0$ such that α satisfies the persistence of excitation condition

$$\mu_1 \leq \int_t^{t+T} |\alpha(s)|^2 \, ds \leq \mu_2, \qquad \forall t \in \mathbb{R}. \tag{1.20}$$

Notice that, with respect to (1.6), the upper bound is also necessary in (1.20) since one assumes here that α takes its values in $\mathbb{R}$ instead of the bounded interval $[0,1]$.

Theorem 1.17 [164, Theorem 8]**.** *Assume that* (A,B) *is controllable and that* $\alpha \in \mathcal{C}^{d-1}(\mathbb{R},\mathbb{R})$ *is bounded, has bounded derivatives up to order* $d-1$*, and satisfies* (1.20) *for some* $T, \mu_1, \mu_2 > 0$*. Let* $P \in \mathrm{GL}_d(\mathbb{R})$ *be such that* (PAP^{-1}, PB) *is in the controllable canonical form*

$$PAP^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_d \end{pmatrix}, \qquad PB = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let $k = \max\left\{2, 2^{\lceil \log_2 d \rceil}\right\}$*. Then there exists* $\lambda_0 > 0$*, depending only on* A*,* B*, and* d*, such that, for every* $\lambda > \lambda_0$*, the feedback control law*

$$u(t) = \sum_{j=2}^{d} a_j \frac{\omega_j(t) - z_j(t)}{\alpha(t)} - \frac{\dot{\omega}_d(t) - \dot{z}_d(t)}{\alpha(t)} - \frac{\alpha(t)^{k-1}}{2r(t)} \omega_d(t) \tag{1.21}$$

renders the system (1.19) *globally exponentially stable, where* $z(t) = Px(t)$*,* r *is the solution of*

$$\begin{cases} \dot{r}(t) = -\lambda r(t) + \alpha(t)^k, \\ r(0) = r_0, \end{cases}$$

with some initial condition $r_0 > 0$, *and* $\omega_1(t), \dots, \omega_d(t)$ *are defined by*

$$\begin{cases} \omega_1(t) = z_1(t), \\ \omega_j(t) = \dot{\omega}_{j-1}(t) + \dfrac{\alpha(t)^k}{2r(t)} \omega_{j-1}(t), \qquad j \in [\![2, d]\!]. \end{cases}$$

Notice that the control law (1.21) is well-defined since, thanks to [164, Remark 6], $\omega_j(t) - z_j(t)$, $j \in [\![2, d]\!]$, and $\dot{\omega}_d(t) - \dot{z}_d(t)$ all depend on $\alpha(t)$ homogeneously, with an homogeneity degree at least one, which means that one can give a canonical meaning to the divisions in (1.21) when the denominator becomes zero (see also [164, Remark 9] for more details). The constant λ_0 is also explicitly characterized in [164, Theorem 8] in terms of only $a_1, \dots, a_d$ and the dimension d. The result of Theorem 1.17 was generalized in [165] to multi-input systems, where one also shows that arbitrary convergence rates can be achieved by such time-dependent feedback laws.

1.2.1.3 Maximal rates of convergence and divergence

As recalled in Proposition 1.4, when considering the linear time-invariant system (1.9), stabilization by linear feedback laws $u = Kx$ at arbitrary rates of convergence, as described in Proposition 1.4(c), is equivalent to destabilization by linear feedback laws $u = Kx$ at arbitrary rates of divergence, as described in Proposition 1.4(d), and both properties are equivalent to the controllability of the pair (A, B) and to the pole-shifting property from Proposition 1.4(b). A natural question, which has been addressed in [45, 49, 53, 128], is whether such properties, or at least some of them, also hold for the persistently excited system (1.7).

Definition 1.18. Let $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ and $T, \mu \in \mathbb{R}_+^*$ with $T \geq \mu$. For $K \in \mathcal{M}_{m,d}(\mathbb{R})$, $x_0 \in \mathbb{R}^d$, and $\alpha \in \mathcal{G}(T, \mu)$, we denote the unique solution of (1.10) with initial condition x_0 by $x(\cdot; x_0, \alpha, A, B, K)$.

(a) For $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $\alpha \in \mathcal{G}(T, \mu)$, we define

$$\lambda^+(\alpha, A, B, K) = \sup_{x_0 \in \mathbb{R}^d \setminus \{0\}} \limsup_{t \to +\infty} \frac{1}{t} \log \lvert x(t; x_0, \alpha, A, B, K) \rvert,$$
$$\lambda^-(\alpha, A, B, K) = \inf_{x_0 \in \mathbb{R}^d \setminus \{0\}} \liminf_{t \to +\infty} \frac{1}{t} \log \lvert x(t; x_0, \alpha, A, B, K) \rvert.$$

(b) For $K \in \mathcal{M}_{m,d}(\mathbb{R})$, the *rate of convergence* and *rate of divergence* of (1.10) are defined respectively by

$$\mathrm{rc}(A, B, K, T, \mu) = -\sup_{\alpha \in \mathcal{G}(T, \mu)} \lambda^+(\alpha, A, B, K), \qquad \mathrm{rd}(A, B, K, T, \mu) = \inf_{\alpha \in \mathcal{G}(T, \mu)} \lambda^-(\alpha, A, B, K).$$

(c) The *maximal rate of convergence* and *maximal rate of divergence* of (1.10) are defined respectively by

$$\mathrm{RC}(A, B, T, \mu) = \sup_{K \in \mathcal{M}_{m,d}(\mathbb{R})} \mathrm{rc}(A, B, K, T, \mu), \qquad \mathrm{RD}(A, B, T, \mu) = \sup_{K \in \mathcal{M}_{m,d}(\mathbb{R})} \mathrm{rd}(A, B, K, T, \mu).$$

Notice that K is a (T, μ)-stabilizer for (1.7) if and only if $\mathrm{rc}(A, B, K, T, \mu) > 0$. Moreover, rc, rd, RC, and RD are all invariant under linear changes of variables, i.e.,

$$\begin{aligned} \mathrm{rc}(A, B, K, T, \mu) &= \mathrm{rc}(PAP^{-1}, PBQ^{-1}, QKP^{-1}, T, \mu), \\ \mathrm{rd}(A, B, K, T, \mu) &= \mathrm{rd}(PAP^{-1}, PBQ^{-1}, QKP^{-1}, T, \mu), \\ \mathrm{RC}(A, B, T, \mu) &= \mathrm{RC}(PAP^{-1}, PBQ^{-1}, T, \mu), \\ \mathrm{RD}(A, B, T, \mu) &= \mathrm{RD}(PAP^{-1}, PBQ^{-1}, T, \mu), \end{aligned} \tag{1.22}$$

for all $P \in \mathrm{GL}_d(\mathbb{R})$, $Q \in \mathrm{GL}_m(\mathbb{R})$. One also obtains immediately that, for every $\lambda \in \mathbb{R}$,

$$\mathrm{RC}(A+\lambda \mathrm{Id}_d, B, T, \mu) = \mathrm{RC}(A, B, T, \mu) - \lambda, \qquad \mathrm{RD}(A+\lambda \mathrm{Id}_d, B, T, \mu) = \mathrm{RD}(A, B, T, \mu) + \lambda, \tag{1.23}$$

and that both $\mathrm{RC}(A,B,T,\mu)$ and $\mathrm{RD}(A,B,T,\mu)$ are non-decreasing functions of μ.

Statements (c) and (d) from Proposition 1.4 can be rephrased simply as $\mathrm{RC}(A,B,T,T) = +\infty$ and $\mathrm{RD}(A,B,T,T) = +\infty$, respectively, and hence Proposition 1.4 states that both these properties are equivalent, and they hold if and only if (A,B) is controllable. A first generalization of this result to the case of persistently excited systems is the following, proved in [49].

Proposition 1.19 [49, Proposition 4.3]. *Let $(A,B) \in \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_{2,1}(\mathbb{R})$ and $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Assume that (A,B) is controllable. Then $\mathrm{RC}(A,B,T,\mu) = +\infty$ if and only if $\mathrm{RD}(A,B,T,\mu) = +\infty$.*

The main ideas of the proof of Proposition 1.19 in [49] are the following. Using (1.22) and (1.23), one first reduces to the case

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for some $a \in \mathbb{R}$. The result is proved by showing that, if $C > 0$ is large enough and $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix} \in \mathcal{M}_{1,2}(\mathbb{R})$ is such that $\mathrm{rc}(A,B,K,T,\mu) > C$, then one has $\mathrm{rd}(A,B,K_-,T,\mu) > C$, where $K_- = \begin{pmatrix} k_1 & -k_2 \end{pmatrix}$. One notices that solutions of $\dot x(t) = (A + \alpha(t) BK_-)x(t)$ can be regarded as solutions of $\dot x(t) = (A + \alpha(t) BK)x(t)$ going backwards in time, in the sense that, for $\tau \in \mathbb{R}_+$ and $t \in [0, \tau]$,

$$x(t; x_0, \alpha, A, B, K_-) = D x(\tau - t; x_1, \alpha(\tau - \cdot), A, B, K), \tag{1.24}$$

where $x_1 = D x(\tau; x_0, \alpha, A, B, K_-)$ and $D = \mathrm{diag}(1, -1)$. This remark allows one to relate the growth a solution of $\dot x(t) = (A + \alpha(t) BK_-)x(t)$ on the interval $[0, \tau]$ to the decay of the corresponding solution of $\dot x(t) = (A + \alpha(\tau - t) BK)x(t)$ according to (1.24). However, such comparison can only be performed on finite time intervals.

The technique of [49] to overcome this difficulty and obtain information on the asymptotic behavior of solutions of $\dot x(t) = (A + \alpha(\tau - t) BK)x(t)$ from (1.24) is to modify α backwards in time on an interval $[-\sigma, 0)$ for some $\sigma \geq 0$, in such a way that $\alpha|_{[-\sigma,\tau]}$ can be extended by periodicity to a (T,μ)-persistently exciting signal $\widetilde{\alpha}$, and such that the projection of the solution of $\dot x(t) = (A + \widetilde{\alpha}(\tau - t) BK)x(t)$ on the unit circle $\mathbb{S}^1$ becomes periodic. Such periodicity of the projected trajectory allows one to obtain information on the asymptotic behavior of the solution only from its decay on a finite time interval corresponding to a period. In order to show that the required modification of α on $[-\sigma, 0)$ can be performed, [49] proves the controllability in finite time of the control system on $\mathbb{S}^1$ obtained by the projection of the control system $\dot x(t) = (A + \xi(t) BK_-)x(t)$, where $\xi(t) \in [0,1]$ is regarded as a control input constrained to satisfy the condition of persistence of excitation (1.6).

The idea of comparing convergence and divergence rates using time reversal and studying a projected system with periodic trajectories has been generalized in [45] to systems in higher dimensions, obtaining the following result.

Theorem 1.20 [45, Theorem 5.4]. *Let $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ and $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Assume that there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the Lie algebra generated by $\{A - \frac{\mathrm{Tr}\, A}{d} \mathrm{Id}_d, BK - \frac{\mathrm{Tr}(BK)}{d} \mathrm{Id}_d\}$ is equal to $\mathfrak{sl}(d, \mathbb{R}) = \{M \in \mathcal{M}_d(\mathbb{R}) \mid \mathrm{Tr}\, M = 0\}$. Then*

$$\mathrm{RC}(A,B,T,\mu) = \mathrm{RD}(-A,-B,T,\mu).$$

The Lie algebraic condition in Theorem 1.20 finds its origin in the study of the control system on the real projective space $\mathbb{RP}^{d-1}$ obtained by the projection of the control system $\dot{x}(t) = (A+\xi(t)BK)x(t)$, where $\xi(t)$ is regarded as a control input. Such condition is equivalent to a Lie algebraic condition for vector fields in $\mathbb{RP}^{d-1}$, and also to a simpler Lie algebraic condition for vector fields $\mathbb{R}^d$ if d is at least three. In the next result, for $M \in \mathcal{M}_d(\mathbb{R})$, ΠM denotes the vector field on $\mathbb{RP}^{d-1}$ obtained by the canonical projection of the vector field $x \mapsto Mx$.

Proposition 1.21 [45, Proposition 5.1]. *Let $(A,B) \in \mathcal{M}_d(\mathbb{R})\times\mathcal{M}_{d,m}(\mathbb{R})$. The following statements are equivalent.*

(a) *There exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the Lie algebra generated by $\{A - \frac{\operatorname{Tr} A}{d}\operatorname{Id}_d, BK - \frac{\operatorname{Tr}(BK)}{d}\operatorname{Id}_d\}$ is equal to $\mathfrak{sl}(d,\mathbb{R})$.*

(b) *There exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for every $q \in \mathbb{RP}^{d-1}$, the evaluation at q of the Lie algebra generated by $\{\Pi A, \Pi BK\}$ is equal to the tangent space $T_q\mathbb{RP}^{d-1}$.*

Moreover, when $d \geq 3$, the above statements are also equivalent to the following one.

(c) *There exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the Lie algebra generated by $\{A, BK\}$ is equal to $\mathcal{M}_d(\mathbb{R})$.*

With respect to Proposition 1.19, Theorem 1.20 replaces the controllability hypothesis on (A,B) by a Lie algebraic condition. A natural question is whether there is any relation between such conditions. This question has been addressed in [45], where it is shown that the Lie algebraic condition from Theorem 1.20 "almost" implies the controllability of (A,B), and that, at least in the single-input case $m = 1$, a converse also holds.

Proposition 1.22 [45, Proposition 5.3 and Theorem 6.1]. *Let $(A,B) \in \mathcal{M}_d(\mathbb{R})\times\mathcal{M}_{d,m}(\mathbb{R})$.*

(a) *If (A,B) is not controllable and there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the Lie algebra generated by $\{A - \frac{\operatorname{Tr} A}{d}\operatorname{Id}_d, BK - \frac{\operatorname{Tr}(BK)}{d}\operatorname{Id}_d\}$ is equal to $\mathfrak{sl}(d,\mathbb{R})$, then $B = 0$, $d = 2$, and the eigenvalues of A are non-real.*

(b) *If (A,B) is controllable and $m = 1$, then there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that the Lie algebra generated by $\{A - \frac{\operatorname{Tr} A}{d}\operatorname{Id}_d, BK - \frac{\operatorname{Tr}(BK)}{d}\operatorname{Id}_d\}$ is equal to $\mathfrak{sl}(d,\mathbb{R})$.*

Thanks to this result, one can also see that Theorem 1.20 provides a generalization of Proposition 1.19. Indeed, if $(A,B) \in \mathcal{M}_2(\mathbb{R})\times\mathcal{M}_{2,1}(\mathbb{R})$ is controllable, Proposition 1.22(b) shows that the Lie algebraic hypothesis of Theorem 1.20 is satisfied. Thanks to (1.23), it suffices to prove Proposition 1.19 for traceless matrices and, when $d = 2$, any traceless matrix A is similar to its opposite $-A$, which shows, using (1.22), that Proposition 1.19 can be obtained from Theorem 1.20.

Proposition 1.19 and Theorem 1.20 consider the relations between the maximal convergence and divergence rates RC and RD, but another interesting question is to characterize the cases where one has $\operatorname{RC}(A,B,T,\mu) = +\infty$ or $\operatorname{RD}(A,B,T,\mu) = +\infty$. For the linear time-invariant system (1.9), which corresponds to taking $\mu = T$ in (1.7), Proposition 1.4 shows that both conditions are equivalent to the controllability of (A,B). However, the situation is different for persistently excited systems, as shown in the following result from [49].

Proposition 1.23 [49, Propositions 4.4 and 4.5].

(a) *Let $d \in \mathbb{N}^*$. There exists $\rho^\star \in (0,1)$ depending only on d such that, for every $T,\mu \in \mathbb{R}_+^*$ with $T \geq \mu$ and $\mu/T > \rho^\star$ and $(A,B) \in \mathcal{M}_d(\mathbb{R})\times\mathcal{M}_{d,1}(\mathbb{R})$ controllable, one has $\operatorname{RC}(A,B,T,\mu) = \operatorname{RD}(A,B,T,\mu) = +\infty$.*

(b) *There exists* $\rho_\star \in (0,1)$ *such that, for every* $T, \mu \in \mathbb{R}_+^*$ *with* $\mu/T < \rho_\star$ *and* $(A,B) \in \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_{2,1}(\mathbb{R})$ *controllable, one has* $\mathrm{RC}(A,B,T,\mu) < +\infty$ *and* $\mathrm{RD}(A,B,T,\mu) < +\infty$.

Notice that Propositions 1.16 and 1.23 are equivalent, due to (1.23) and to the fact that, if (A,B) is controllable, then $(A+\lambda\,\mathrm{Id}_d, B)$ is also controllable for every $\lambda \in \mathbb{R}$.

The proof of Proposition 1.23(b) given in [49] provides some insight on the origin of the phenomenon of non-stabilizability at arbitrary rate of convergence. Indeed, the idea of such proof is to actually construct, for some $\lambda \in \mathbb{R}$ and for each feedback matrix $K \in \mathcal{M}_{1,2}(\mathbb{R})$, a (T,μ)-signal α which destabilizes the system $\dot{x} = (A + \lambda\,\mathrm{Id}_d + \alpha(t) BK) x(t)$. This construction exploits the overshoot phenomenon that happens when switching between systems $\dot{x} = Ax$ and $\dot{x} = (A+BK)x$, which consists on the fact that the norm of a solution of an asymptotically stable system may increase before decreasing, a fact also used in Example 1.1, for instance. Hence, switching after the increase of the norm and before its decrease can have a destabilizing effect. It is also interesting to note that the overshoot prevents stabilization in the case where μ/T is small, but not for μ/T large. The signal α constructed in such proof takes its values on $\{0,1\}$, is periodic, and oscillates faster between 0 and 1 as K increases in norm.

This technique of proof led to the conjecture, formulated in [49], that, if one imposes additional constraints on the signal α preventing fast switching, it might be possible to recover stabilizability at arbitrary rates of convergence. This conjecture was first proved to be true for two-dimensional systems in [128], by considering Lipschitz continuous signals α.

Theorem 1.24 [128, Theorem 3.1]**.** *Let* $(A,B) \in \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_{2,1}(\mathbb{R})$ *be controllable and* $T, \mu, M \in \mathbb{R}_+^*$ *be such that* $T \geq \mu$. *Then, for every* $\gamma > 0$, *there exists* $K \in \mathcal{M}_{1,2}(\mathbb{R})$ *and* $C > 0$ *such that, for every* $\alpha \in \mathcal{G}(T,\mu)$ *Lipschitz continuous with Lipschitz constant* M, *every solution* x *of* (1.10) *satisfies*

$$|x(t)| \leq C e^{-\gamma t} |x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

The proof of this result relies on the planar dynamics and cannot be directly generalized to higher dimensions. The time is separated into "good" time intervals, where the feedback is sufficiently active in order to stabilize the system, and "bad" time intervals, where the feedback is not enough active and an explosive behavior may occur. Such explosive behavior is due not only to the dynamics of A, but it may also come from the dynamics of $A + \alpha BK$ when α is too small. A technique of worst-case trajectory, similar to that of [20, 32, 125], is used to analyze the maximal rate of explosion on "bad" time intervals, showing that it is compensated by the convergence on "good" ones.

Arbitrary rate of convergence can also be retrieved if one assumes that the persistently exciting signal α has a uniformly bounded total variation on bounded time intervals and takes values in $\{0,1\}$, as shown in the following result from [46].

Theorem 1.25 [46, Theorem 4.3]**.** *Let* $(A,B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ *be controllable and* $T, \mu, M \in \mathbb{R}_+^*$ *be such that* $T \geq \mu$. *Then, for every* $\gamma > 0$, *there exists* $K \in \mathcal{M}_{m,d}(\mathbb{R})$ *and* $C > 0$ *such that, for every* $\alpha \in \mathcal{G}(T,\mu)$ *taking values in* $\{0,1\}$ *and with total variation on* $[t, t+T]$ *bounded by* M *for every* $t \in \mathbb{R}$, *every solution* x *of* (1.10) *satisfies*

$$|x(t)| \leq C e^{-\gamma t} |x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

A key point in the proof of Theorem 1.25 is that, if (A,B) is controllable and $\gamma \geq 1$, one can choose a feedback matrix K such that the constant C from Proposition 1.4(c) depends *polynomially* on γ, a fact which is proved, e.g., in [42, 43], with improved bounds provided in [99]. This allows one to estimate the overshoot on time intervals where α is active but not for long enough in order to induce a decrease of the norm of the solution, and show that it can be counteracted by the decrease of the solution on time intervals where α is active for longer.

Another situation where one can also obtain stabilizability at an arbitrary rate of convergence is when $\operatorname{rk} B = d$ (see [46, Theorem 4.4]).

All the previous results concerning stabilizability at an arbitrary rate of convergence require the stabilizability of the persistently excited system (1.7) for *all* persistently exciting signals α in $\mathcal{G}(T,\mu)$ or in a subset of $\mathcal{G}(T,\mu)$. In particular, (1.10) should be stable even for the *worst* possible signals α, that is, those α which give the slowest decay rates. However, as highlighted in the proof of [49, Proposition 4.5], or also on related works on switched systems [20, 32], such worst trajectories are typically very specific, corresponding to very fast switching or to switching at very precise times. It is natural to imagine that, in practical situations, such specific behavior is very unlikely to occur, and that the typical practical behavior should be much better than the worst theoretical behavior.

These ideas motivate the study of the stabilizability of (1.7) where, instead of considering that α satisfies the persistence of excitation condition (1.6) and trying to stabilize the system for all $\alpha \in \mathcal{G}(T,\mu)$ as in Definitions 1.7 and 1.18, one assumes that α is generated by a certain random process, which, as (1.6), ensures that the control u is active often enough, and considers the stabilizability of (1.7) for almost all such signals α. The intuition is that, under some rather mild hypotheses on the random process generating α, one should avoid the particular situations impairing stability in Proposition 1.16(b) and recover stabilization at arbitrary convergence rates. Such study has been carried out in [53] and is the subject of Chapter 2.

We consider, in Chapter 2, the more general framework of a switched system with N subsystems and with a random switching signal obtained from a discrete Markov chain, driving the switches between the subsystems, and from probability laws on $(0,+\infty)$ with finite expected value, defining the time spent on each subsystem. We characterize its Lyapunov exponents by applying the Multiplicative Ergodic Theorem to an associated discrete-time system, and, as an application of such characterization, we prove that a controllability condition for a switched control system implies that arbitrary exponential decay rates for almost sure stabilization can be obtained by linear feedback laws (see Theorem 2.36 and also Remark 2.38 for the relation with the persistence of excitation condition (1.6)).

1.2.2 Infinite-dimensional persistently excited systems

The work presented in Chapters 3 and 4 of this thesis were motivated by the study of persistently excited systems in infinite dimension. Even though infinite-dimensional switched systems have attracted much research effort in the past few years [7, 79, 92, 111, 124, 149], very few works have considered persistently excited systems in infinite dimension [47, 91]. This section presents the most important results of [91, 92], which considers the generalization of Theorem 1.9 and Proposition 1.10 to infinite-dimensional systems. The paper [47] is the subject of Chapter 3.

For $T \geq \mu > 0$, consider the persistently excited control system

$$\dot{z}(t) = Az(t) + \alpha(t)Bu(t), \qquad z(t) \in \mathsf{H},\ u(t) \in \mathsf{U},\ \alpha \in \mathcal{G}(T,\mu), \tag{1.25}$$

where H, U are Hilbert spaces, the linear operator $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, and $B \in \mathcal{L}(\mathsf{U},\mathsf{H})$. As in Section 1.2.1, we are interested in asymptotically stabilizing (1.25) by means of a linear feedback $u(t) = Kz(t)$ for some bounded operator $K \in \mathcal{L}(\mathsf{H},\mathsf{U})$. Notice that, for every $K \in \mathcal{L}(\mathsf{H},\mathsf{U})$, $\alpha \in \mathcal{G}(T,\mu)$, and $z_0 \in \mathsf{H}$, the closed-loop system

$$\dot{z}(t) = (A + \alpha(t)BK)z(t) \tag{1.26}$$

admits a unique *mild* solution $z \in \mathcal{C}(\mathbb{R}_+, \mathsf{H})$ (see, e.g., [21]), i.e., z is the unique function in $\mathcal{C}(\mathbb{R}_+, \mathsf{H})$ satisfying, for every $t \geq 0$,

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}\alpha(s)BKz(s)ds.$$

The paper [91] has considered the problem of whether (1.25) can be asymptotically stabilized by the linear feedback $u(t) = -B^*z(t)$ when A generates a strongly continuous *contraction* semigroup $\{e^{tA}\}_{t\geq 0}$, i.e., $\left\|e^{tA}\right\|_{\mathcal{L}(\mathsf{H})} \leq 1$ for every $t \geq 0$. The interesting case is when the uncontrolled evolution $\dot{z}(t) = Az(t)$ does not generate a strict contraction, i.e., when $\left\|e^{tA}\right\| = 1$ for every $t \geq 0$, so that the norm of solutions may remain constant in the absence of control. Notice that an immediate generalization of Proposition 1.10 to infinite-dimensional systems does not hold, as shown in the following example.

Example 1.26 [91, Example 2.1]**.** Let us consider a damped wave equation on a string of unitary length with fixed endpoints, whose dynamics are described by

$$\begin{cases} \partial_{tt}^2 v(t,x) = \partial_{xx}^2 v(t,x) - \alpha(t)\zeta(x)^2\partial_t v(t,x), & t \in [0,+\infty),\ x \in [0,1], \\ v(0,x) = v_0(x), & x \in [0,1], \\ \partial_t v(0,x) = v_1(x), & x \in [0,1], \\ v(t,0) = v(t,1) = 0, & t \in [0,+\infty), \end{cases} \tag{1.27}$$

where $\zeta \in L^\infty((0,1),\mathbb{R})$ and $\alpha \in L^\infty(\mathbb{R}_+,[0,1])$. This can be written under the form (1.26) by setting the real Hilbert spaces H and U to be $\mathsf{H} = H_0^1((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, $\mathsf{U} = L^2((0,1),\mathbb{R})$, with the usual scalar product in $L^2((0,1),\mathbb{R})$ and the scalar product in $H_0^1((0,1),\mathbb{R})$ defined by $\langle v,w\rangle_{H_0^1((0,1),\mathbb{R})} = \langle \partial_x v, \partial_x w\rangle_{L^2((0,1),\mathbb{R})}$. We define the operators A and B by

$$D(A) = \left(H^2((0,1),\mathbb{R}) \cap H_0^1((0,1),\mathbb{R})\right) \times H_0^1((0,1),\mathbb{R}),$$
$$A = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}, \qquad \text{i.e.,} \quad A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \frac{d^2 z_1}{dx^2} \end{pmatrix},$$
$$B = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}, \qquad \text{i.e.,} \quad Bu = \begin{pmatrix} 0 \\ \zeta u \end{pmatrix},$$

we set $z = (v, \partial_t v)$ and take $K = -B^*$.

A straightforward computation shows that $D(A^*) \supset D(A)$ and that A^* and $-A$ coincide in $D(A)$. Since A is surjective, it follows that $A^* = -A$, and thus A is skew-adjoint. By Stone's theorem (see, for instance, [170, Theorem 3.8.6]), A generates a strongly continuous unitary group $\{e^{At}\}_{t\in\mathbb{R}}$, and, in particular, $\left\|e^{tA}\right\| = 1$ for every $t \in \mathbb{R}$. Notice also that, if ζ is not identically zero, the control system $\dot{z}(t) = Az(t) + Bu(t)$ is exactly controllable in time greater than 2 (see, e.g., [55, Theorem 2.55]).

However, in general, (1.27) is not asymptotically stable. Indeed, assume that $\zeta = \chi_{(a,b)}$ is the characteristic function of a proper subinterval $(a,b) \subsetneq (0,1)$, where we assume, without loss of generality, that $b < 1$. Then there exist $T, \mu \in \mathbb{R}_+^*$ with $T \geq \mu$, a persistently exciting signal $\alpha \in \mathcal{G}(T,\mu)$, and a corresponding nonzero periodic solution of (1.27). This follows from the results in [124] (see also [92]) and can be illustrated by an explicit counterexample. Set $b' = \frac{1+b}{2}$. Take $T = 2$ and $\mu = 1 - b'$. Then

$$\alpha = \sum_{k=0}^{\infty} \chi_{[2k-\mu,2k+\mu)} \tag{1.28}$$

is a (T,μ)-persistently exciting signal and

$$v(t,x) = \sum_{k=0}^{\infty} (\chi_{[b'+2k,1+2k]}(x+t) - \chi_{[-1-2k,-b'-2k]}(x-t)) \tag{1.29}$$

is a periodic, nonzero, mild solution of (1.27) corresponding to the signal α. Notice, in particular, that this solution does not converge to zero, even in the weak sense. We also remark that one can replace the characteristic functions in (1.29) by translations of a smooth function in order to obtain a smooth solution for (1.27).

Example 1.26 shows that Proposition 1.10 cannot be immediately generalized to infinite-dimensional persistently excited systems. The paper [91] provide extra conditions under which one can guarantee the asymptotic stability of

$$\dot{z}(t) = (A - \alpha(t)BB^*)z(t). \tag{1.30}$$

A first such result proves that exponential stability holds if a generalized observability inequality is satisfied.

Theorem 1.27 [91, Theorem 3.2]**.** *Let $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ be the generator of a strongly continuous contraction semigroup, $B \in \mathcal{L}(\mathsf{U},\mathsf{H})$, and $T,\mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Suppose there exist two constants $c,\tau > 0$ such that*

$$\int_0^{\tau} \alpha(t) \left\| B^* e^{tA} z_0 \right\|_{\mathsf{U}}^2 dt \geq c \|z_0\|_{\mathsf{H}}^2, \qquad \forall z_0 \in \mathsf{H},\ \alpha \in \mathcal{G}(T,\mu). \tag{1.31}$$

Then there exist $C \geq 1$ and $\gamma > 0$ such that, for every initial condition $z_0 \in \mathsf{H}$ and $\alpha \in \mathcal{G}(T,\mu)$, the corresponding solution z of (1.30) satisfies

$$\|z(t)\|_{\mathsf{H}} \leq Ce^{-\gamma t} \|z_0\|_{\mathsf{H}}, \qquad \forall t \in \mathbb{R}_+. \tag{1.32}$$

The proof of Theorem 1.27 relies on the study of the Lyapunov function $V(z) = \frac{1}{2}\|z\|_{\mathsf{H}}^2$, for which the following estimate can be shown.

Lemma 1.28 [91, Lemma 2.1]**.** *Let $a,b \in \mathbb{R}_+$ be such that $b \geq a$. Then, for every measurable function $\alpha : \mathbb{R}_+ \to [0,1]$, every mild solution z of (1.30) satisfies*

$$V(z(b)) - V(z(a)) \leq -\frac{1}{2 + 2(b-a)^2 \|B\|^4} \int_0^{b-a} \alpha(t+a) \left\| B^* e^{tA} z(a) \right\|_{\mathsf{U}}^2 dt.$$

Thanks to Lemma 1.28, one can provide a proof for Theorem 1.27.

Proof of Theorem 1.27. Fix $\alpha \in \mathcal{G}(T,\mu)$ and $s \geq 0$, and let $V(z) = \frac{1}{2}\|z\|_{\mathsf{H}}^2$. Lemma 1.28 with $a = s$ and $b = s + \tau$ yields

$$V(z(s+\tau)) - V(z(s)) \leq -\frac{1}{2\left(1 + \tau^2 \|B\|^4\right)} \int_0^{\tau} \alpha(t+s) \left\| B^* e^{tA} z(s) \right\|_{\mathsf{U}}^2 dt,$$

and so (1.31) implies

$$V(z(s+\tau)) - V(z(s)) \leq -\frac{c}{1 + \tau^2 \|B\|^4} V(z(s)).$$

The desired estimate (1.32) follows from standard arguments. ∎

Notice that (1.31) is a generalization of the usual observability inequality for the exact controllability of the system $\dot{z}(t) = A^* z(t) + Bu(t)$ (see, e.g., [55, Theorem 2.42]). In particular, by taking α to be constant and equal to 1 in (1.31), one obtains that a necessary condition for (1.31) to hold is the exact controllability of $\dot{z}(t) = A^* z(t) + Bu(t)$ in some time $\tau > 0$. It has been proved in [91, Example 3.1] that the generalized observability inequality (1.31) holds, for instance, for the wave equation

$$\begin{cases} \partial_{tt} v(t,x) = \Delta v(t,x) - \alpha(t)\zeta(x)^2 \partial_t v(t,x), & t \in [0,\infty),\ x \in \Omega, \\ \quad v(0,x) = v_0(x), & x \in \Omega, \\ \partial_t v(0,x) = v_1(x), & x \in \Omega, \\ \quad v(t,x) = 0, & t \in [0,\infty),\ x \in \partial\Omega, \end{cases} \tag{1.33}$$

where Ω is a bounded domain of $\mathbb{R}^d$ and $\zeta \in L^\infty(\Omega, \mathbb{R})$ satisfies $|\zeta(x)| \geq \zeta_0$ for some $\zeta_0 > 0$ and almost every $x \in \Omega$. This result has been proved in [91] by means of a spectral decomposition of the Dirichlet Laplacian, with $\tau = T$.

Another stability result for (1.30) proved in [91] is the following, which shows that a generalized unique continuation property is a sufficient condition for weak asymptotic stability of (1.30).

Theorem 1.29 [91, Theorem 3.2]**.** *Let $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ be the generator of a strongly continuous contraction semigroup, $B \in \mathcal{L}(\mathsf{U}, \mathsf{H})$, and $T, \mu \in \mathbb{R}_+^*$ be such that $T \geq \mu$. Suppose there exists $\tau > 0$ such that, for every $\alpha \in \mathcal{G}(T, \mu)$ and $z_0 \in \mathsf{H}$,*

$$\int_0^\tau \alpha(t) \left\lVert B^* e^{At} z_0 \right\rVert_{\mathsf{U}}^2 dt = 0 \qquad \Longrightarrow \qquad z_0 = 0. \tag{1.34}$$

Then, for every $z_0 \in \mathsf{H}$ and $\alpha \in \mathcal{G}(T, \mu)$, the corresponding solution z of (1.30) converges weakly to 0 in H as $t \to +\infty$.

Theorem 1.29 is proved by first showing that, for every $z_0 \in \mathsf{H}$ and $\alpha \in \mathcal{G}(T, \mu)$, the *weak ω-limit set*

$$\begin{aligned} \omega(z_0, \alpha) = \{ z_\infty \in \mathsf{H} \mid\ & \text{there exists a sequence } (s_n)_{n \in \mathbb{N}} \text{ with } s_n \to +\infty \text{ such that the solution} \\ & z \text{ of (1.30) associated with } z_0 \text{ and } \alpha \text{ satisfies } z(s_n) \rightharpoonup z_\infty \text{ as } n \to +\infty \} \end{aligned}$$

is non-empty. This follows from the fact that the norm of a solution decreases along trajectories, since A generates a contraction semigroup, and so any trajectory admits a weak limit point. The main part of the proof consists on establishing that, if $z_\infty \in \omega(z_0, \alpha)$, then there exists $\alpha_\infty \in \mathcal{G}(T, \mu)$ such that

$$\int_0^\tau \alpha_\infty(t) \left\lVert B^* e^{At} z_\infty \right\rVert_{\mathsf{U}}^2 dt = 0, \tag{1.35}$$

and thus the assertion of the theorem follows from (1.34). We refer to [91] for the detailed proof of (1.35).

Similarly to (1.31), (1.34) is a generalization of the usual unique continuation property for the approximate controllability of the system $\dot{z}(t) = A^* z(t) + Bu(t)$ (see, e.g., [55, Theorem 2.43]), and, in particular, a necessary condition for (1.34) to hold is the approximate controllability of $\dot{z}(t) = A^* z(t) + Bu(t)$ in some time $\tau > 0$. According to [91, Example 4.1], the generalized unique continuation property (1.34) holds, for instance, for the Schrödinger equation

$$\begin{cases} i\partial_t v(t,x) = -\Delta v(t,x) - i\alpha(t)\zeta(x)^2 v(t,x), & t \in [0,\infty),\ x \in \Omega, \\ \quad v(0,x) = v_0(x), & x \in \Omega, \\ \quad v(t,x) = 0, & t \in [0,\infty),\ x \in \partial\Omega, \end{cases} \tag{1.36}$$

where Ω is a bounded domain of $\mathbb{R}^d$ and $\zeta \in L^\infty(\Omega, \mathbb{R})$ satisfies $|\zeta(x)| \geq \zeta_0$ for some $\zeta_0 > 0$ and almost every $x \in \omega$ for some non-empty open set $\omega \subset \Omega$. This result has been proved in [91] by a spectral decomposition of the Dirichlet Laplacian and a combination of Privalov's and Holmgren's uniqueness theorems (see, e.g., [107, Chapter III, Section D] or [177, Chapter XIV, Theorem 1.9] for the former and [96, Theorem 8.6.8] for the latter), in a technique similar to that of [150].

Another problem treated in [91] is to obtain sufficient conditions for the asymptotic stability of (1.30) when α is not a persistently exciting signal, but satisfies instead some other condition guaranteeing a persistent action of the control on the system. One such condition is the following.

Definition 1.30 [91, Definition 5.1]. Let $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ be the generator of a strongly continuous contraction semigroup, $B \in \mathcal{L}(\mathsf{U}, \mathsf{H})$, and $T, c > 0$. The set of all signals $\alpha \in L^\infty([0,T],[0,1])$ satisfying

$$\int_0^T \alpha(t) \left\|B^* e^{tA} z_0\right\|_{\mathsf{U}}^2 dt \geq c \|z_0\|_{\mathsf{H}}^2, \quad \forall z_0 \in \mathsf{H} \tag{1.37}$$

is denoted by $\mathcal{K}(A, B, T, c)$.

One of the stability results presented in [91] using this definition is the following criterion for the strong convergence to zero of solutions of (1.30).

Theorem 1.31 [91, Theorem 5.3]. *Let $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ be the generator of a strongly continuous contraction semigroup and $B \in \mathcal{L}(\mathsf{U}, \mathsf{H})$. Suppose that there exist $\rho, T_0 \in \mathbb{R}_+^*$ and a continuous function $c : (0,\infty) \to (0,\infty)$ such that, for all $T \in (0, T_0]$, if $\tilde{\alpha} \in L^\infty([0,T],[0,1])$ is such that $\int_0^T \tilde{\alpha}(t) dt \geq \rho T$, then $\tilde{\alpha} \in \mathcal{K}(A, B, T, c(T))$.*

Let $((a_n, b_n))_{n\in\mathbb{N}}$ be a sequence of disjoint intervals in $\mathbb{R}_+$ with $\sum_{n\in\mathbb{N}} c(b_n - a_n) = +\infty$ and $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ be such that

$$\int_{a_n}^{b_n} \alpha(t) dt \geq \rho(b_n - a_n), \qquad \forall n \in \mathbb{N}. \tag{1.38}$$

Then any mild solution z of (1.30) satisfies $\|z(t)\|_{\mathsf{H}} \to 0$ as $t \to +\infty$.

Notice that (1.38) is a generalization of the persistence of excitation condition (1.6), where one requires a lower bound on the integral of α not on all intervals of length T, but only on a sequence of intervals $((a_n, b_n))_{n\in\mathbb{N}}$ which do not become too small too fast, in the sense that $\sum_{n\in\mathbb{N}} c(b_n - a_n) = +\infty$.

Even if the hypotheses of Theorem 1.31 are quite technical, it has been shown in [91, Example 5.2] that it can be applied to the wave equation (1.33), where, as before, one assumes that $\zeta \in L^\infty(\Omega, \mathbb{R})$ satisfies $|\zeta(x)| \geq \zeta_0$ for some $\zeta_0 > 0$ and almost every $x \in \Omega$, but without the assumption that α is persistently exciting. In this case, the sufficient condition for the strong convergence to zero of solutions of (1.30) obtained from Theorem 1.31 in [91, Example 5.2] is that there exists $\rho > 0$ and a sequence of disjoint intervals $((a_n, b_n))_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} (b_n - a_n)^3 = +\infty$ such that $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ satisfies (1.38). As in the previous cases, this result is also shown using spectral methods.

Another consequence of Theorem 1.31 is the following improvement of Proposition 1.10 for finite-dimensional systems, where one no longer assumes α to be persistently exciting.

Proposition 1.32 [91, Corollary 5.5]. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$. Assume that A is skew-symmetric and (A, B) is controllable, and let $r \in \mathbb{N}$ be the smallest non-negative integer such that $\operatorname{rk}\begin{pmatrix} B & AB & \cdots & A^r B\end{pmatrix} = d$. Let $\rho > 0$, $((a_n, b_n))_{n\in\mathbb{N}}$ be a sequence of disjoint intervals in $\mathbb{R}_+$ with $\sum_{n=1}^\infty (b_n - a_n)^{2r+1} = +\infty$, and $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ be such that (1.38) is satisfied. Then every solution of (1.30) satisfies $|z(t)| \to 0$ as $t \to +\infty$.*

The stability of infinite-dimensional systems more general than (1.30) with time-varying damping parameters satisfying some condition guaranteeing a persistent action on the system have also been considered in other works in the literature. For instance, [92] analyzes the stability of the second order system

$$\begin{cases} \ddot{z}(t) + B(t)\dot{z}(t) + Az(t) = 0, \\ z(0) = z_0 \in \mathsf{V}, \\ \dot{z}(0) = z_1 \in \mathsf{H}, \end{cases} \tag{1.39}$$

where $A : D(A) \subset \mathsf{H} \to \mathsf{H}$ is a linear self-adjoint coercive operator with dense domain, V denotes the Hilbert space $\mathsf{V} = D(A^{1/2})$, and the time-dependent (and a priori unbounded and non-linear) operator B satisfies $B(t)0 = 0$ and $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{Lip}(\mathsf{W}, \mathsf{W}'))$, where $\mathrm{Lip}(\mathsf{W}, \mathsf{W}')$ denotes the set of Lipschitz continuous functions from the Hilbert space W to its dual W'. One also assumes that $\mathsf{V} \hookrightarrow \mathsf{W} \hookrightarrow \mathsf{H} \equiv \mathsf{H}' \hookrightarrow \mathsf{W}' \hookrightarrow \mathsf{V}'$ with dense embeddings, and that there exist $C, C_0, \lambda_0 > 0$ and $b \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ such that, for every $\lambda \in [0, \lambda_0]$, $(\mathrm{Id}_{\mathsf{W}} + \lambda A)^{-1} \in \mathcal{L}(\mathsf{W})$ with

$$\left\|(\mathrm{Id}_{\mathsf{W}} + \lambda A)^{-1}\right\|_{\mathcal{L}(\mathsf{W})} \leq C_0,$$

and, for every $t \in \mathbb{R}_+$ and $v, w \in \mathsf{W}$,

$$\begin{aligned} \langle B(t)w - B(t)v, w - v\rangle_{\mathsf{W}',\mathsf{W}} &\geq 0, \\ \langle B(t)w, w\rangle_{\mathsf{W}',\mathsf{W}} &\geq b(t)\|w\|_{\mathsf{W}}^2, \\ \|B(t)w - B(t)v\|_{\mathsf{W}'} &\leq Cb(t)\|w - v\|_{\mathsf{W}}. \end{aligned}$$

Under these assumptions, the existence and uniqueness of the solutions of (1.39) have been proved in [92, Theorem 2.1] in the function space $L^2((0,T), \mathsf{V}) \cap H^1((0,T), \mathsf{H}) \cap H^2((0,T), \mathsf{V}')$, proving that solutions belong to the space $\mathcal{C}^0([0,T], \mathsf{V}) \cap \mathcal{C}^1([0,T], \mathsf{H})$. The main result of [92] is the following stability criterion, which relies on an estimate of the energy decay on short intervals of time established in [92, Theorem 3.1].

Theorem 1.33 [92, Theorem 3.2]**.** *Consider System (1.39) with the previous assumptions. Let $((a_n, b_n))_{n\in\mathbb{N}}$ be a sequence of non-empty disjoint open intervals in $(0, +\infty)$. Assume that there exist sequences $(m_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying*

$$\sum_{n=0}^{+\infty} m_n(b_n - a_n)\min\left((b_n - a_n)^2, \frac{M_n}{M_n + m_n}\right) = +\infty,$$

and such that, for every $t \in (a_n, b_n)$ and $v \in \mathsf{W}$, one has

$$m_n\|v\|_{\mathsf{W}}^2 \leq \langle B(t)v, v\rangle_{\mathsf{W}',\mathsf{W}} \leq M_n\|B(t)v\|_{\mathsf{W}'}^2.$$

Then every solution z of (1.39) satisfies $\|z(t)\|_{\mathsf{V}} \to 0$ and $\|\dot{z}(t)\|_{\mathsf{H}} \to 0$ as $t \to +\infty$.

The previous results from [91, 92] consist in a great contribution for the study of persistently excited and switched systems in infinite dimension. However, several problems concerning the infinite-dimensional persistently excited system (1.25) remain open, such as the case of control laws other than $u(t) = -B^* z(t)$, unbounded control operators B, operators A for which e^{tA} is not necessarily a contraction, or dynamics on Banach spaces.

Motivated by the several open problems on infinite-dimensional persistently excited systems, this thesis analyzes the behavior of one such system in Chapter 3. More precisely, we are interested in the asymptotic behavior of the system of transport equations

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(x)u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t, L_j), & t \geq 0,\ i \in [\![1, N]\!], \\ u_i(0, x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!], \end{cases} \tag{1.40}$$

where, for $i \in [\![1, N]\!]$, χ_i is the characteristic function of an interval $[a_i, b_i] \subset [0, L_i]$ with $a_i < b_i$, α_i is a persistently exciting signal, and $M = (m_{ij})_{i,j\in[\![1,N]\!]} \in \mathcal{M}_N(\mathbb{R})$ is called the *transmission matrix*. The main result of Chapter 3 is Theorem 3.1, which provides sufficient conditions for the exponential stability of (1.40).

The analysis carried out in Chapter 3 does not use the results from [91], since one is interested in studying (1.40) in the Banach space $\prod_{i=1}^N L^p([0, L_i], \mathbb{R})$ for $p \in [1, +\infty]$, which is not a Hilbert space unless $p = 2$, and, even in that case, the operator A associated with (1.40) may not be a contraction. We rely rather on an explicit formula for the solutions of (1.40), expressing the solution at time t in terms of the initial condition and some coefficients computed recursively. The stability analysis is performed by studying the behavior of such coefficients. The application of such technique to more general problems is also the main motivation for Chapter 4.

It is also interesting to note that the stability result obtained in Chapter 3 cannot be obtained from a result similar to Theorem 1.27 using a generalized observability inequality, since Theorem 3.1 in Chapter 3 guarantees the stability for some situations where it is known that such generalized observability inequality does not hold. Indeed, consider the case $N = 2$, $N_d = 1$, $L_1/L_2 \notin \mathbb{Q}$, and $m_{11} = m_{12} = m_{21} = m_{22} = \frac{1}{2}$, in which exponential stability of (1.40) is guaranteed by Theorem 3.1. If $u_{1,0}(x) = \varphi(L_1 - x)$ and $u_{2,0}(x) = -\varphi(L_2 - x)$ for some $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ compactly supported in $(0, \delta)$ for some small $\delta > 0$, then one immediately obtains that the solution of the undamped system

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in \{1, 2\}, \\ u_i(t, 0) = \dfrac{1}{2}(u_1(t, L_1) + u_2(t, L_2)), & t \geq 0,\ i \in \{1, 2\}, \\ u_i(0, x) = u_{i,0}(x), & x \in [0, L_i],\ i \in \{1, 2\}, \end{cases}$$

satisfies $u_i(t, x) = 0$ for every $i \in \{1, 2\}$, $x \in [0, L_i]$, and $t > \delta$. Hence, if $T - \mu > \delta$, one can always find non-zero initial conditions and some $\alpha \in \mathcal{G}(T, \mu)$ for which the left-hand side of (1.31) is zero, which proves that (1.31) cannot be satisfied.

1.3 Systems of partial differential equations on networks

The study of System (1.40) carried out in Chapter 3 is motivated by the several open problems on infinite-dimensional persistently excited systems, but also by the fact that (1.40) can be seen a simple case of a *multi-body* or *multi-link structure*. These type of problems model strings, membranes, or plates, by partial differential equations defined on several coupled domains, and are an active research subject since the 1980s [4, 5, 119, 120, 137, 138]. Such research activity is motivated by the applications of multi-body structures and the interesting mathematical questions that arise from their analysis (see, e.g., [6, 110] and references therein). The particular case of (1.40) can be seen as a system of transport equations on a network, whose edges are identified with the N intervals $[0, L_i]$ for $i \in [\![1, N]\!]$, the connection of such edges being described by the matrix M. Notice that the case considered in Chapter 3, where one assumes $m_{ij} \neq 0$ for every $i, j \in [\![1, N]\!]$ (see Section 3.2.3 and Hypothesis 3.11), corresponds to a network containing a single node O from where all the edges start and end.

Systems of partial differential equations on networks are a particular kind of multi-body systems which have attracted much research effort recently [35, 63]. Such systems are modeled by several PDEs on one-dimensional domains, each domain being identified with an edge of a given graph, with interactions between the PDEs occurring at the vertices of the graph. Despite the simplification provided by the one-dimensional dynamics on each edge, the interactions on the vertices render the analysis of such systems far from trivial. For instance, several properties of systems of wave equations on networks depend on the topology of the network or on rationality relations of the lengths of its edges [48, 63]. Consider, for instance, the following system of wave equations on a network

$$\begin{cases} \dfrac{\partial^2 u_e}{\partial t^2}(t,x) = \dfrac{\partial^2 u_e}{\partial x^2}(t,x), & e \in \mathcal{E},\ t \in [0,+\infty),\ x \in [0, L_e], \\ u_{e_1}(t,q) = u_{e_2}(t,q), & q \in \mathcal{V},\ e_1, e_2 \in \mathcal{E}_q,\ t \in [0,+\infty), \\ \displaystyle\sum_{e \in \mathcal{E}_q} \dfrac{\partial u_e}{\partial n_e}(t,q) = 0, & q \in \mathcal{V}_{\mathrm{int}},\ t \in [0,+\infty), \end{cases} \tag{1.41}$$

with either Dirichlet controls

$$u_e(t,q) = v_q(t), \qquad q \in \mathcal{V}_{\mathrm{ext}},\ e \in \mathcal{E}_q,\ t \in [0,+\infty), \tag{1.42}$$

or Neumann controls

$$\frac{\partial u_e}{\partial t}(t,q) = v_q(t), \qquad q \in \mathcal{V}_{\mathrm{ext}},\ e \in \mathcal{E}_q,\ t \in [0,+\infty), \tag{1.43}$$

where $\mathcal{N}$ is a connected network, $\mathcal{E}$ it its set of edges, $\mathcal{V}$ is its set of vertices, $L_e > 0$ is the length of the edge e, $\mathcal{E}_q$ denotes the set of edges containing the vertex $q \in \mathcal{V}$, $\mathcal{V}_{\mathrm{int}}$ is the set of all vertices of $\mathcal{N}$ belonging to at least two different edges, called *interior* vertices, $\mathcal{V}_{\mathrm{ext}}$ is the set of all vertices of $\mathcal{N}$ belonging to only one edge, called *exterior* vertices, and, for $q \in \mathcal{V}_{\mathrm{ext}}$, v_q are control inputs. The following result from [63] highlights the dependence of the controllability of (1.41) on the rationality relations of the lengths of the edges.

Theorem 1.34 [63, Corollary 5.38]. *Consider the system of wave equations (1.41) with Dirichlet controls (1.42). Assume that $\mathcal{V}_{\mathrm{int}}$ contains only one node and that there exists $q_0 \in \mathcal{V}_{\mathrm{ext}}$ such that $v_q = 0$ for every $q \in \mathcal{V}_{\mathrm{int}} \setminus \{q_0\}$. Let e_0 be the only edge in $\mathcal{E}_{q_0}$ and set $L_{\mathcal{N}} = \sum_{e \in \mathcal{E}} L_e$. Then (1.41) is approximately controllable in some time $T \geq 2L_{\mathcal{N}}$ if and only if $\frac{L_{e_1}}{L_{e_2}} \notin \mathbb{Q}$ for every $e_1, e_2 \in \mathcal{E} \setminus \{e_0\}$ with $e_1 \neq e_2$.*

Theorem 1.34 illustrates the fact that the rationality relations of the lengths of the edges have an influence in the behavior of the system. Several other results exist where such dependency is more subtle; for more details, we refer to [62, 63, 171]. Another property that influences the behavior of (1.41) is the topology of the network, which is illustrated, for instance, in the following stabilization result, whose proof is given in Chapter 4, Theorem 4.65, in a more general setting.

Theorem 1.35 [48, Theorem 5.16]. *Consider the system of wave equations (1.41) with Neumann controls (1.43) given by $v_q(t) = -\eta_q \frac{\partial u_e}{\partial n_e}(t,q)$ for $q \in \mathcal{V}_{\mathrm{ext}}$, where $\eta_q \in [0,+\infty)$ is a damping coefficient. Then (1.41) is exponentially stable if and only if $\mathcal{N}$ is a tree and $\eta_q = 0$ for at most one $q \in \mathcal{V}_{\mathrm{ext}}$.*

We refer to Chapter 4 for a more precise definition of systems of wave equations on networks and the sense in which one considers their solutions.

Several different systems of partial differential equations on networks have been considered in the literature, such as systems of Euler–Bernoulli beam equations [8, 130, 157], usually motivated by problems in mechanics; of wave equations [2, 25, 62, 63, 139, 176] or conservations laws [24, 145], usually motivated by propagation phenomena; or of Schrödinger equations [26, 98], motivated for instance by the applications to the study of quantum graphs [3, 108, 109]. Several works analyze only some simple network topologies, such as *star-shaped* networks (i.e., networks with a central vertex belonging to all edges, as in Theorem 1.34) [62, 79] or *tree-shaped* networks (i.e., networks without cycles) [2, 26, 98, 145], but, despite such simplification, the dynamics in these cases are sufficiently rich to present several interesting phenomena due to the network structure, and their study is still of much mathematical and practical interest. Notice that the system of transport equations (1.40) studied in Chapter 3 is defined on a star-shaped network.

The main technique used in the study of (1.40) in Chapter 3 is an explicit formula for its solutions (see Theorem 3.18), obtained using the method of characteristics and an iterative argument. The main idea for retrieving such formula is the following. Let $(u_1, \dots, u_N)$ be a solution of (1.40), assumed here to be sufficiently regular (see Section 3.2.1 for the well-posedness of (1.40)). Notice that, for $i \in [\![1, N]\!]$, $x \in [0, L_i]$, and $t \geq x$, one has, by the method of characteristics, that

$$u_i(t,x) = u_i(t-x,0) e^{-\int_0^x \alpha_i(t-s)\chi_i(x-s) ds},$$

where we set $\chi_i \equiv 0$ for $i \in [\![N_d+1, N]\!]$, with a similar equation expressing $u_i(t,x)$ in terms of the initial condition $u_{i,0}$ when $0 \leq t < x$. In particular, $u_i(t,x)$ can be computed if one knows $u_i(t,0)$ for $t \geq 0$. Using the third equation of (1.40), one obtains that, for $t \geq L_{\max}$,

$$u_i(t,0) = \sum_{j=1}^N m_{ij} u_j(t-L_j, 0) e^{-\int_0^{L_j} \alpha_j(t-s)\chi_j(L_j-s) ds}. \tag{1.44}$$

Such equation expresses $u_i(t,0)$ in terms of $u_j(\cdot, 0)$ evaluated at previous times for $j \in [\![1, N]\!]$. The explicit formula for the solution of (1.40) is obtained in Chapter 3 by iterating (1.44) in order to express $u_i(t,0)$ in terms of the initial conditions $u_{j,0}$, $j \in [\![1, N]\!]$.

Notice that, by setting $v(t) = (u_i(t,0))_{i \in [\![1,N]\!]} \in \mathbb{R}^N$, one obtains that $v(t)$ satisfies

$$v(t) = \sum_{j=1}^N A_j(t) v(t-L_j), \tag{1.45}$$

where $A_j(t) \in \mathcal{M}_N(\mathbb{R})$ is given by $A_j(t) = \left(a_{k\ell}^{(j)}(t)\right)_{k,\ell \in [\![1,N]\!]}$ with

$$a_{k\ell}^{(j)}(t) = \begin{cases} m_{kj} e^{-\int_0^{L_j} \alpha_j(t-s)\chi_j(L_j-s) ds}, & \text{if } \ell = j, \\ 0, & \text{if } \ell \neq j. \end{cases}$$

The techniques used in Chapter 3 turn out to be also applicable to analyze the stability of more general systems under the form (1.45). Moreover, other systems of hyperbolic equations on networks, more general than (1.40), can also be put under the form (1.45), such as linear wave equations on networks (see Section 4.4 for more details). This motivates the study of systems of the form (1.45), known as *difference equations*, which is the main subject of Chapters 4 and 5 of this thesis.

1.4 Difference equations

Consider the system

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j), \qquad t \geq 0, \tag{1.46}$$

where $x(t) \in \mathbb{C}^d$, and, for $j \in [\![1, N]\!]$, $A_j \in \mathcal{M}_d(\mathbb{C})$ and $\Lambda_j > 0$ is a delay. System (1.46) is called an *autonomous difference equation* and its analysis has attracted much interest since the 1970s [14, 60, 64, 84, 94, 129] (see also [86, Chapter 9] and references therein) and to this day [48, 87, 127, 132]. The well-posedness of such system can be easily established in several different function spaces (see Sections 4.2.1 and 5.2), such as the Lebesgue spaces L^p, the Sobolev spaces $W^{k,p}$, or the $\mathcal{C}^k$ spaces, possibly with compatibility conditions being prescribed in order to ensure the desired regularity. In order to fix the ideas, we consider in this introduction the Banach spaces X_0 and X defined by

$$\begin{aligned} \mathsf{X} &= \mathcal{C}([-r, 0], \mathbb{C}^d), \\ \mathsf{X}_0 &= \left\{ x \in \mathsf{X} \;\middle|\; x(0) = \sum_{j=1}^{N} A_j x(-\Lambda_j) \right\}, \end{aligned}$$

with the usual L^∞ norm, where $r \geq \Lambda_{\max}$. For every initial condition $x_0 \in \mathsf{X}_0$, there exists a unique $x \in \mathcal{C}([-r, +\infty), \mathbb{C}^d)$ satisfying (1.46) and such that $x(t) = x_0(t)$ for $t \in [-\Lambda_{\max}, 0]$ and $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0]} \in \mathsf{X}_0$ for every $t \geq 0$, the map $t \mapsto x_t \in \mathsf{X}_0$ being continuous (see, e.g., Proposition 5.2 and Remark 5.4 in Chapter 5, or also [86, Chapter 9, Theorem 1.1 and Lemma 2.1]). Notice that one may also consider (1.46) with $x(t) \in \mathbb{R}^d$ and $A_j \in \mathcal{M}_d(\mathbb{R})$, but we choose complex-valued states and matrices in this section following the approach of [48, 127], used in Chapters 4 and 5 below, which is motivated by the fact that classical results for difference equations are more naturally written down in such framework.

1.4.1 Stability of difference equations

The stability of (1.46) has been studied through spectral methods and Laplace transform techniques, leading to several stability criteria, such as the following one from [60, 86, 94].

Theorem 1.36 [86, Chapter 9, Theorem 3.5]**.** *The following statements are equivalent.*

(a) *System (1.46) is uniformly asymptotically stable in* X_0.

(b) *There exist* $C, \gamma > 0$ *such that, for every* $x_0 \in \mathsf{X}_0$, *the solution* x *of (1.46) with initial condition* x_0 *satisfies*

$$\lvert x(t) \rvert \leq C e^{-\gamma t} \lVert x_0 \rVert_{\mathsf{X}_0}, \qquad \forall t \in \mathbb{R}_+.$$

(c) *One has*

$$\sup \left\{ \operatorname{Re} \lambda \;\middle|\; \lambda \in \mathbb{C},\ \det \left(\mathrm{Id}_d - \sum_{j=1}^{N} e^{-\lambda \Lambda_j} A_j \right) = 0 \right\} < 0. \tag{1.47}$$

In practical applications, the coefficients of the matrices A_j and the delays Λ_j may be known only up to a certain precision, and it is thus important to known the effects that small perturbations have in the behavior of (1.46). The fact that the stability of (1.46) is preserved under small perturbations of the matrices A_j can be easily established from (1.47) (see, e.g., [86, Section 9.6]). However, the left-hand side of (1.47) is not continuous in general with respect to the delays $\Lambda_1, \dots, \Lambda_N$, which means that the stability of (1.46) is not preserved under small perturbations of the delays, a fact that has been already remarked in [94, 129, 134] and that we illustrate with the following example, which is adapted from [86, Section 9.6].

Example 1.37. Consider the difference equation

$$x(t) = -x(t-1) - \frac{1}{2}x(t-2), \qquad t \geq 0. \tag{1.48}$$

In order to study its stability using Theorem 1.36, we study the zeroes of the characteristic equation $1 + e^{-\lambda} + \frac{1}{2}e^{-2\lambda} = 0$. A straightforward computation shows that such equation is satisfied if and only if $e^{\lambda} = -\frac{1}{2} + \frac{i}{2}$ or $e^{\lambda} = -\frac{1}{2} - \frac{i}{2}$. In particular, for every $\lambda \in \mathbb{C}$ solution of $1 + e^{-\lambda} + \frac{1}{2}e^{-2\lambda} = 0$, one has $e^{2 \operatorname{Re} \lambda} = \left|e^{\lambda}\right|^2 = \frac{1}{2}$, which yields $\operatorname{Re} \lambda = -\frac{1}{2}\log 2$. Hence (1.47) is satisfied, and, by Theorem 1.36, (1.48) is uniformly asymptotically stable.

Consider now, for $n \in \mathbb{N}^*$, the difference equation

$$x(t) = -x\left(t - \frac{4n+2}{4n-1}\right) - \frac{1}{2}x(t-2), \qquad t \geq 0. \tag{1.49}$$

Notice that $\frac{4n+2}{4n-1} \to 1$ as $n \to +\infty$, and hence, for large n, (1.49) is a perturbation of (1.48). Let $\sigma_n \in \mathbb{R}_+^*$ be the unique positive real number satisfying $e^{-\frac{4n+2}{4n-1}\sigma_n} + \frac{1}{2}e^{-2\sigma_n} = 1$. Notice that σ_n is well-defined, since the function $f_n : \mathbb{R}_+ \to \mathbb{R}$ given by $f_n(\sigma) = e^{-\frac{4n+2}{4n-1}\sigma} + \frac{1}{2}e^{-2\sigma} - 1$ satisfies $f_n(0) = \frac{1}{2}$, $\lim_{\sigma \to +\infty} f_n(\sigma) = -1$, and $f_n^\prime(\sigma) < 0$ for every $\sigma > 0$. Moreover, one has $f_n(\sigma) \geq \frac{3}{2}e^{-2\sigma} - 1$ for every $n \in \mathbb{N}^*$ and $\sigma \in \mathbb{R}_+$, since $\frac{4n+2}{4n-1} \leq 2$. In particular, $f_n\left(\frac{1}{10}\right) \geq \frac{3}{2}e^{-1/5} - 1 > 0$, which proves that $\sigma_n > \frac{1}{10}$ for every $n \in \mathbb{N}^*$. Let $\omega_n = \frac{4n-1}{2}\pi$ and consider the function

$$x(t) = e^{\sigma_n t}\sin(\omega_n t).$$

A straightforward computation shows that x is a solution of (1.49). Hence, for every $n \in \mathbb{N}^*$, (1.49) is unstable, admitting an unbounded solution x which grows faster than $e^{t/10}$.

Example 1.37 shows that small perturbations on the delays may drastically change the stability of (1.46). An important question is, therefore, to characterize the situations where the stability of (1.46) is preserved under perturbations on the delays. For that purpose, the following definition has been introduced in [86–88].

Definition 1.38 [86, Chapter 9, Definitions 6.1 and 6.2]**.** Let $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$ and $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$.

(a) System (1.46) is said to be *locally strongly stable* if there exists a neighborhood V of $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ in $(0, +\infty)^N$ such that, for every $L = (L_1, \dots, L_N) \in V$, the system

$$x(t) = \sum_{j=1}^{N} A_j x(t - L_j), \qquad t \geq 0, \tag{1.50}$$

is uniformly asymptotically stable.

(b) System (1.46) is said to be *globally strongly stable* (or simply *strongly stable*) if, for every $L = (L_1, \dots, L_N) \in (0, +\infty)^N$, (1.50) is uniformly asymptotically stable.

The following result, known as the Hale–Silkowski criterion, characterizes the strong stability of (1.46).

Theorem 1.39 [14, Theorem 5.2]. *Let $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$. The following assertions are equivalent.*

(a) *One has $\rho_{\mathrm{HS}}(A) < 1$, where*

$$\rho_{\mathrm{HS}}(A) = \max_{(\theta_1,\dots,\theta_N)\in[0,2\pi]^N} \rho\left(\sum_{j=1}^{N} e^{i\theta_j} A_j\right). \tag{1.51}$$

(b) *There exist $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$ rationally independent such that (1.46) is uniformly asymptotically stable.*

(c) *There exist $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$ such that (1.46) is locally strongly stable.*

(d) *System (1.46) is globally strongly stable.*

Hence, global and local strong stability are equivalent, a striking fact first proved in [82]. Moreover, they are equivalent to the uniform asymptotic stability of (1.46) for fixed $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$ with rationally independent components, and can be characterized by the condition $\rho_{\mathrm{HS}}(A) < 1$, a fact first proved in [159]. Theorem 1.39 was already known to hold in the one-dimensional case since [129]. Another interesting feature of Theorem 1.39 is that $\rho_{\mathrm{HS}}(A)$ does not depend on the delays $\Lambda_1, \dots, \Lambda_N$. This result has been generalized in [132] to the case where one assumes to have some rational dependence structure on the delays, which is important since, in some practical situations, the delays cannot be chosen independently. The stability of (1.46) with time-varying matrices A_j has been considered in [48, 136] and is the subject of Chapter 4. Stability issues for time-varying delays Λ_j have been addressed in [15].

1.4.2 Neutral functional differential equations

A major motivation for analyzing the stability of (1.46) is that it is deeply related to properties of more general neutral functional differential equations of the form

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{N} A_j x(t - \Lambda_j)\right] = L x_t, \tag{1.52}$$

where $x_t : [-r, 0] \to \mathbb{C}^d$ is given by $x_t(s) = x(t+s)$, $r \geq \max_{j\in\{1,\dots,N\}} \Lambda_j$, and $L : \mathsf{X} \to \mathbb{C}^d$ is a bounded linear map. Notice that (1.52) can also be written as

$$x(t) - \sum_{j=1}^{N} A_j x(t - \Lambda_j) = x(0) - \sum_{j=1}^{N} A_j x(-\Lambda_j) + \int_0^t L x_s \, ds,$$

highlighting the link between (1.46) and (1.52). For every $x_0 \in \mathsf{X}$, (1.52) admits a unique solution $x \in \mathcal{C}([-r, +\infty), \mathbb{C}^d)$ satisfying $x(t) = x_0(t)$ for $t \in [-r, 0]$ (see, e.g., [86, Chapter 9, Theorem 1.1]). The fact that the analysis of (1.46) can provide information on (1.52) is illustrated by the following result, which provides some properties of (1.52) that can be obtained when (1.46) is strongly stable.

Theorem 1.40 [86, Chapter 9, Theorems 7.1 and 7.3]. *Let $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$, $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$, and assume that (1.46) is strongly stable. Let $x_0 \in \mathsf{X}$ and denote by $\gamma^+(x_0), \omega(x_0) \subset \mathsf{X}$, the sets*

$$\gamma^+(x_0) = \{x_t \mid x \text{ is the solution of (1.52) with initial condition } x_0,\ t \geq 0\},$$

$$\omega(x_0) = \bigcap_{s\geq 0} \overline{\{x_t \mid x \text{ is the solution of } (1.52) \text{ with initial condition } x_0,\ t \geq s\}},$$

called respectively the positive orbit *and the* ω-limit set *of* x_0.

(a) *The positive orbit* $\gamma^+(x_0)$ *is relatively compact if and only if it is bounded.*

(b) *If* $\gamma^+(x_0)$ *is bounded, then* $\omega(x_0)$ *is a nonempty, compact, connected, analytic, invariant set.*

(c) *If* $x \in \mathcal{C}((-\infty, a], \mathbb{C}^d)$ *is a solution of* (1.52), *then it is analytic.*

Some conclusions of Theorem 1.40 also hold when the linear operator L is replaced by a continuous function $f : \Omega \to \mathbb{C}^d$ for some open subset Ω of X (see [86, Section 9.7]).

One can also provide relations on the spectra of the strongly continuous semigroups associated with (1.46) and (1.52) [94]. For $t \geq 0$, let $T_0(t) \in \mathcal{L}(\mathsf{X}_0)$ be the operator given by $T_0(t)x_0 = x_t$, where x is the unique solution of (1.46) with initial condition x_0, and define $T(t) \in \mathcal{L}(\mathsf{X})$ by $T(t)x_0 = x_t$, where x denotes now the unique solution of (1.52) with initial condition x_0. Then $\{T_0(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ are strongly continuous semigroups in X_0 and X, respectively, whose generators correspond to transport operators [85]. Notice that, since $L : \mathsf{X} \to \mathbb{C}^d$ is a bounded linear operator, it follows by Riesz representation theorem that there exists a function of bounded variation $\eta : [-r, 0] \to \mathcal{M}_d(\mathbb{C})$ such that, for every $x \in \mathsf{X}$,

$$Lx = \int_{-r}^{0} d\eta(s) x(s), \tag{1.53}$$

where we also denote by η the Lebesgue–Stieltjes measure associated with the function of bounded variation η.

Recall that, for a closed linear operator $A : D(A) \subset \mathsf{Y} \to \mathsf{Y}$ in a Banach space Y, its *spectrum* $\sigma(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not bijective. It is decomposed into the *point spectrum* $\sigma_{\mathrm{p}}(A)$, which is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective, the *residual spectrum* $\sigma_{\mathrm{r}}(A)$, which is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective but its range is not dense in Y, and the *continuous spectrum* $\sigma_{\mathrm{c}}(A)$, which is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective and its range is dense in Y but not equal to Y. Hence $\sigma(A) = \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{r}}(A) \cup \sigma_{\mathrm{c}}(A)$, the unions being pairwise disjoint. The *essential spectrum* $\sigma_{\mathrm{ess}}(A)$ of A is the part of the spectrum of A that cannot be removed by compact perturbations of A, i.e., $\sigma_{\mathrm{ess}}(A) = \bigcap_{K \text{ compact}} \sigma(A+K)$. When $\sigma(A)$ is non-empty, the *spectral radius* of A is defined by $\rho(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$, the spectral radii $\rho_{\mathrm{p}}(A)$, $\rho_{\mathrm{r}}(A)$, $\rho_{\mathrm{c}}(A)$, and $\rho_{\mathrm{ess}}(A)$ being defined similarly.

The next theorem, which gathers several results from [94] (see also [87]), provides several links between the spectra of (1.46) and (1.52).

Theorem 1.41 [94]. *Let* $t \geq 0$ *and set*

$$Z_0 = \left\{ \lambda \in \mathbb{C} \;\middle|\; \det\left[\mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda\Lambda_j}\right] = 0 \right\},$$

$$Z = \left\{ \lambda \in \mathbb{C} \;\middle|\; \det\left[\lambda\left(\mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda\Lambda_j}\right) - \int_{-r}^{0} e^{\lambda s} d\eta(s)\right] = 0 \right\}.$$

(a) *The point spectra of* $T_0(t)$ *and* $T(t)$ *satisfy*

$$\sigma_{\mathrm{p}}(T_0(t)) \setminus \{0\} = \left\{ e^{\lambda t} \;\middle|\; \lambda \in Z_0 \right\}, \qquad \sigma_{\mathrm{p}}(T(t)) \setminus \{0\} = \left\{ e^{\lambda t} \;\middle|\; \lambda \in Z \right\}.$$

(b) *The residual spectra of* $T_0(t)$ *and* $T(t)$ *are both empty.*

(c) *The continuous spectra of $T_0(t)$ and $T(t)$ satisfy*

$$\sigma_c(T_0(t)) \setminus \{0\} \subset \left\{\lambda \in \mathbb{C} \,\middle|\, |\lambda| = e^{\mu t},\ \mu \in \overline{\operatorname{Re} Z_0}\right\}, \quad \sigma_c(T(t)) \setminus \{0\} \subset \left\{\lambda \in \mathbb{C} \,\middle|\, |\lambda| = e^{\mu t},\ \mu \in \overline{\operatorname{Re} Z_0}\right\}.$$

(d) *The essential spectra of $T_0(t)$ and $T(t)$ coincide outside* 0*, i.e.,* $\sigma_{\mathrm{ess}}(T(t))\setminus\{0\} = \sigma_{\mathrm{ess}}(T_0(t))\setminus\{0\}$.

(e) *The spectral radius of $T_0(t)$ is given by* $\rho(T_0(t)) = \rho_{\mathrm{ess}}(T_0(t)) = \rho_{\mathrm{ess}}(T(t)) = e^{\alpha t}$*, where* $\alpha = \sup \operatorname{Re} Z_0$.

(f) *The spectral radius of $T(t)$ is given by* $\rho(T(t)) = \rho_{\mathrm{p}}(T(t)) = e^{\beta t}$*, where* $\beta = \sup \operatorname{Re} Z$.

Notice that Theorem 1.41(c) provides the same bound on the continuous spectra of $T_0(t)$ and $T(t)$, which depends only on the set Z_0, associated with the difference equation (1.46). Moreover, thanks to Theorem 1.41(e), the exponential stability of (1.46) is a necessary condition for the exponential stability of the solutions of (1.52).

1.4.3 Hyperbolic partial differential equations

Another important motivation for the study of (1.46) is that several systems of hyperbolic partial differential equations can be put under the form (1.46) or (1.52). This standard approach to the analysis of hyperbolic PDEs relies on the method of characteristics and has been widely used in the literature, since at least the 1960s [33, 34, 54, 74, 133, 160] and to this day [48, 56, 57, 70, 79, 106]. The following example, extracted from [160], exhibits such transformation for a hyperbolic system stemming from an electric circuit connected by a transmission line.

Example 1.42 [160]**.** Consider the electric circuit from Figure 1.2, where a voltage source of voltage E with internal resistance r_0 is connected by a lossless transmission line of unit length to a load, composed of a capacitor of capacitance C connected in parallel to a nonlinear element described by the function f relating the voltage and the current across this element, which usually models a tunnel diode. The voltage v and current i along the transmission line are determined by the system of telegrapher's equations

$$\left\{\begin{aligned} L_0 \partial_t i(t,x) &= -\partial_x v(t,x), && t \geq 0,\ x \in [0,1], \\ C_0 \partial_t v(t,x) &= -\partial_x i(t,x), && t \geq 0,\ x \in [0,1], \\ v(t,0) + r_0 i(t,0) &= E, && t \geq 0, \\ i(t,1) - C\partial_t v(t,1) &= f(v(t,1)), && t \geq 0, \end{aligned}\right. \tag{1.54}$$

where L_0 and C_0 are the specific inductance and capacitance of the transmission line, respectively.

Let $Z = \sqrt{\frac{L_0}{C_0}}$ be the characteristic impedance and $c = \frac{1}{\sqrt{L_0 C_0}}$ the propagation speed of waves on the transmission line. For regular solutions of (1.54), one has that

$$\frac{d}{dt}[v(t, x+ct) + Zi(t, x+ct)] = 0 \quad \text{and} \quad \frac{d}{dt}[v(t, x-ct) - Zi(t, x-ct)] = 0,$$

which proves that solutions of (1.54) must be of the form

$$v(t,x) = \frac{1}{2}[\phi(x-ct) + \psi(x+ct)], \qquad i(t,x) = \frac{1}{2Z}[\phi(x-ct) - \psi(x+ct)], \tag{1.55}$$

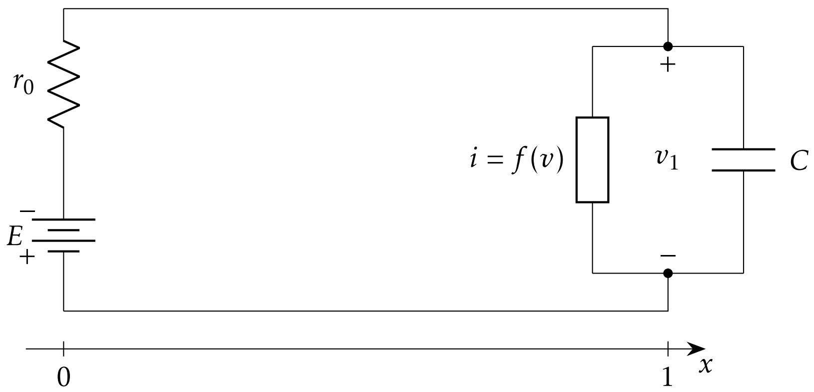


Figure 1.2: Electric circuit from Example 1.42. The generator on the left is connected to the load on the right by a lossless transmission line of unit length.

for some regular functions $\phi : (-\infty, 1] \to \mathbb{C}$ and $\psi : [0, +\infty) \to \mathbb{C}$. Using this transformation, [160] shows that the voltage $v_1(t) = v(t, 1)$ on the load satisfies the equation

$$C \frac{d}{dt}[v_1(t) + \rho v_1(t-\tau)] = \frac{2E}{Z + r_0} - \frac{1}{Z}[v_1(t) - \rho v_1(t-\tau)] - f(v_1(t)) - \rho f(v_1(t-\tau)), \tag{1.56}$$

where $\tau = 2/c$ is the time a wave takes to travel twice the length of the transmission line and $\rho = \frac{r_0 - Z}{r_0 + Z}$ is the reflection coefficient at the extremity $x = 0$. Equation (1.56) is a neutral functional differential equation, being a non-linear generalization of (1.52). Moreover, (1.56) is a non-linear generalization of (1.46) when the load is made only of the non-linear element f, i.e., when $C = 0$, reducing to (1.46) when f is linear.

Notice that it suffices to study (1.56) in order to obtain the behavior of (1.54), in the sense that, if one knows the solution $v_1(t)$ of (1.56) for every $t \geq 0$, then it is possible to reconstruct the solution of (1.54) for every $t \geq 1/c$. Indeed, thanks to the last equation of (1.54), one can obtain $i(t, 1)$ from $v(t, 1)$ for every $t \geq 0$, and hence, using (1.55), one can obtain $\phi(\xi)$ for every $\xi \in (-\infty, 1]$ and $\psi(\xi)$ for every $\xi \in [1, +\infty)$. Another application of (1.55) allows one to reconstruct $v(t, x)$ and $i(t, x)$ for every $x \in [0, 1]$ and $t \geq 1/c$. In particular, the asymptotic behaviors of (1.54) and (1.56) can be obtained one from the other.

Motivated by the previous literature on the difference equation (1.46) and its applications, we analyze, in Chapter 4, the stability of the non-autonomous difference equation

$$x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \tag{1.57}$$

where $A_j : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})$ for $j \in [\![1, N]\!]$. Thanks to a suitable representation formula for its solutions, generalizing the one obtained in Chapter 3 for (1.40), we characterize the exponential behavior of (1.57) in terms of some time-dependent matrix coefficients, taking into account the rational dependence structure of the delays $\Lambda_1, \dots, \Lambda_N$. We also provide a generalization of the Hale–Silkowski criterion, Corollaries 4.31 and 4.37 below, characterizing the exponential stability of (1.57) uniformly with respect to $A = (A_1, \dots, A_N) \in L^\infty(\mathbb{R}, \mathfrak{B})$ for some non-empty bounded set $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$. Notice that this situation corresponds to regarding (1.57) as a switched system under arbitrary $\mathfrak{B}$-valued switching signals. By exploiting the link between (1.57) and linear hyperbolic PDEs with time-varying coefficients, we apply our results to the stability analysis of systems of transport and wave propagation on networks, obtaining in particular a characterization of the stability of systems of wave equations on networks with switching damping at exterior vertices in terms of the topology of the network and the number of damped vertices, which generalizes a known result for constant damping.

1.4.4 Control of difference equations

Difference equations and neutral functional difference equations have also been considered in the literature from the point of view of control and stabilization [87,88,140,141,143,154]. In such situations, one is interested in the controlled difference equation

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t) \tag{1.58}$$

or in the controlled neutral functional differential equation

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{N} A_j x(t - \Lambda_j)\right] = Lx_t + Bu(t), \tag{1.59}$$

where $u(t) \in \mathbb{C}^m$ is the control input and $B \in \mathcal{M}_{d,m}(\mathbb{C})$. The stabilizability of (1.58) by a linear feedback law of the form $u(t) = \sum_{j=1}^{N} K_j x(t-\Lambda_j)$ has been addressed in [87], where the following result is shown.

Theorem 1.43 [87, Theorem 3.1]. *Let $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)^N$. The following assertions are equivalent.*

(a) *There exist $K_1, \dots, K_N \in \mathcal{M}_{m,d}(\mathbb{C})$ such that the system*

$$x(t) = \sum_{j=1}^{N} (A_j + BK_j) x(t - \Lambda_j)$$

is strongly stable.

(b) *For every $L_1, \dots, L_N \in (0, +\infty)$, there exists $\varepsilon > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq -\varepsilon$, one has*

$$\operatorname{rk}\begin{pmatrix} B & \mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda L_j}\end{pmatrix} = d. \tag{1.60}$$

Notice that (1.60) is a reminiscent of Hautus test for controllability (see, e.g., [163, Lemma 3.3.7]). The stabilizability of (1.59) by a linear feedback law of the form $u(t) = \frac{d}{dt}\left[\sum_{j=1}^{N} K_j x(t - \Lambda_j)\right] + Gx_t$, where $K_j \in \mathcal{M}_{m,d}(\mathbb{C})$ for $j \in [\![1, N]\!]$ and $G \in \mathcal{L}(\mathsf{X}, \mathbb{C}^m)$, has also been addressed in [87].

Theorem 1.44 [87, Theorem 3.2]. *Let $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $L \in \mathcal{L}(\mathsf{X}, \mathbb{C}^d)$, and $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)^N$, and write L as in (1.53). The following assertions are equivalent.*

(a) *There exist $K_1, \dots, K_N \in \mathcal{M}_{m,d}(\mathbb{C})$ and $G \in \mathcal{L}(\mathsf{X}, \mathbb{C}^m)$ such that the system*

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^{N} (A_j + BK_j) x(t - \Lambda_j)\right] = (L + BG)x_t$$

is strongly stable.

(b) *For every $L_1, \dots, L_N \in (0, +\infty)$, there exists $\varepsilon > 0$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq -\varepsilon$, one has*

$$\operatorname{rk}\begin{pmatrix} B & \lambda\left(\mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda L_j}\right) - \int_{-r}^{0} e^{\lambda s} d\eta(s) \end{pmatrix} = d,$$
$$\operatorname{rk}\begin{pmatrix} B & \mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda L_j} \end{pmatrix} = d.$$

The controllability problem for (1.58) and (1.59) is also of much interest. Notice that, since the dynamics of such equations are infinite-dimensional, taking place in the Banach spaces X_0 and X, respectively, several different notions of controllability can be used, such as exact, approximate, spectral, or relative controllability [51, 154].

Relative controllability consists in controlling only the final state $x(T) \in \mathbb{C}^d$, instead of the whole state $x_T = x(T + \cdot)$ in X_0 or X. This notion has been originally introduced in the study of control systems with delays in the input [19, 51, 105, 142], having been later extended to systems with delays in the state [66, 148] and also to more general frameworks, such as stochastic control systems [103] or fractional integro-differential systems [18]. The relative controllability of (1.58) in a particular situation has been considered in [148], where the following theorem, generalizing a result from [66], is shown.

Theorem 1.45 [148, Theorem 4]**.** *Consider the difference equation*

$$x(t) = x(t-1) + Ax(t-\Lambda) + Bu(t), \tag{1.61}$$

where $\Lambda \in \mathbb{N}^*$, $A \in \mathcal{M}_d(\mathbb{C})$, and $B \in \mathcal{M}_{d,m}(\mathbb{C})$. Assume that $\operatorname{rk} B = m \in [\![1, d]\!]$. Let $T \in \mathbb{N}$. Then the following assertions are equivalent.

(a) *For every $x_0 : [-\Lambda, 0) \to \mathbb{C}^d$ and $x_1 \in \mathbb{C}^d$, there exists $u : [0, T] \to \mathbb{C}^m$ such that the unique solution x of (1.61) with initial condition x_0 and control u satisfies $x(T) = x_1$.*

(b) *One has $T \geq T_{\min}$ and*

$$\operatorname{rk}\begin{pmatrix} B & AB & A^2B & \cdots & A^qB \end{pmatrix} = d,$$

where $T_{\min} = \left\lceil \frac{d}{m} - 1 \right\rceil \Lambda$ and $q = \frac{T_{\min}}{\Lambda} = \left\lceil \frac{d}{m} - 1 \right\rceil$.

Other notions of controllability for (1.58) and (1.59) are less present in the literature, with the remarkable exception of [141, 154] and references therein.

Chapter 5 considers the controllability problem for (1.58). We provide relative controllability criteria in some different function spaces, generalizing the criterion from Theorem 1.45 to the general situation of (1.58) in Theorems 5.12 and 5.13. We also compare relative controllability for different delays in terms of their rational dependence relations and characterize the minimal time for relative controllability. Chapter 5 also considers the exact and approximate controllability of (1.58), showing some general results for commensurable delays, in which case exact and approximate controllability are equivalent, before completely characterizing exact and approximate controllability of (1.58) for a two-dimensional system with two delays and one control input.

1.5 Structure of the thesis

This thesis presents the work carried out in the following articles.

[47] Y. Chitour, G. Mazanti, and M. Sigalotti. Persistently damped transport on a network of circles. *Trans. Amer. Math. Soc.*, to appear.

[48] Y. Chitour, G. Mazanti, and M. Sigalotti. Stability of non-autonomous difference equations with applications to transport and wave propagation on networks. *Netw. Heterog. Media*, to appear.

[53] F. Colonius and G. Mazanti. Lyapunov exponents for random continuous-time switched systems and stabilizability. Preprint arXiv: 1511.06461, 2015.

[127] G. Mazanti. Relative controllability of linear difference equations. Preprint arXiv: 1604.08663, 2016.

Chapter 2 presents the work from [53]. Motivated by the previous study of finite-dimensional persistently excited systems recalled in Section 1.2.1, we investigate the asymptotic behavior of a switched system in continuous time with random switching signals in terms of its Lyapunov exponents. After characterizing such exponents in Theorem 2.31 and providing a formula for the largest Lyapunov exponent in Corollary 2.35, we apply these results to show the stabilizability of a control system with arbitrary rate of convergence in Section 2.6.

Chapter 3 contains the work carried out in [47]. We analyze the exponential stability of (1.40), which is a system of N transport equations with intermittent damping on a network. Such network can be identified with several circles intersecting at a single point O, the coupling between the N equations being a linear mixing of their values at O, described by the transmission matrix M. The activity of the intermittent damping is determined by persistently exciting signals, all belonging to a fixed class $\mathcal{G}(T, \mu)$. The main result is Theorem 3.1, which proves that, under suitable hypotheses on M and on the rationality of the ratios between the lengths of the circles, such system is exponentially stable, uniformly with respect to the persistently exciting signals. The proof relies on an explicit formula for the solutions of this system, which allows one to track down the effects of the intermittent damping.

The technique used in the proof of Theorem 3.1 was generalized in [48] to the stability analysis of non-autonomous difference equations of the form (1.57), and the corresponding results are presented in Chapter 4. We provide a suitable representation of their solutions in terms of their initial conditions and some time-dependent matrix coefficients, generalizing the technique used in Chapter 3. This enables us to characterize the asymptotic behavior of solutions in terms of such matrix coefficients. In the case of difference equations with arbitrary switching, we obtain a delay-independent generalization of Hale–Silkowski stability criterion. Using the classical transformations of hyperbolic PDEs into difference equations, we apply our results to transport and wave propagation on networks, obtaining, as a consequence, that exponential stability of such systems is robust with respect to variations of the lengths of the network edges preserving their rational dependence structure. We then prove that the wave equation on a network with arbitrarily switching damping at external vertices is exponentially stable if and only if the network is a tree and the damping is bounded away from zero at all external vertices but at most one.

Finally, Chapter 5 considers the controllability of the difference equation (1.58). We first consider the relative controllability of (1.58), presenting the work from [127]. This is done by using a suitable formula for the solutions of such systems in terms of their initial conditions, their control inputs, and some matrix-valued coefficients obtained recursively from the matrices defining the system, which is a version of the representation formula from Chapter 4 adapted to the case of the control system (1.58). Thanks to such formula, we characterize relative controllability in time T in terms of an algebraic property of the matrix-valued coefficients, which reduces to the usual Kalman controllability criterion in the case of a single delay, and also generalizes Theorem 1.45. Relative controllability is studied for solutions in the set of all functions and in the function spaces L^p and $\mathcal{C}^k$. We also compare the relative controllability of the system for different delays in terms of their rational dependence structure, proving that relative controllability for some delays implies relative controllability for all delays that are "less rationally dependent" than the original ones, in a sense that we make precise. Moreover, we provide an upper bound on the minimal controllability time for a system depending only on its dimension and on its largest delay.

Chapter 5 also presents, in Secion 5.4, some results on the exact and approximate controllability of (1.58) in L^2. We first consider the case of commensurable delays, in which exact and approximate controllability are equivalent. We prove that a Kalman condition obtained by an augmentation of the state of the system coincides with another controllability criterion obtained from the explicit formula for the solutions of (1.58). We then turn to the case of a two-dimensional system with two delays and one control, for which we completely characterize exact and approximate controllability in Theorem 5.51.

A summary of the main results of this thesis in French is provided in Annexe A.

Chapter 2

Lyapunov exponents for random continuous-time switched systems and applications to the stabilizability of control systems

2.1 Introduction

Linear systems with switching coefficients are of considerable interest in theory and applications. The present chapter considers systems in continuous time with random switching and develops methods to describe the exponential growth rates, i.e., the Lyapunov exponents. This is used to analyze stabilizability properties of control systems with random switching.

Systems with deterministic switching have been extensively studied, cf., e.g., the monograph [113] and the surveys [114, 158]. An important motivation for our work comes from the theory of persistently excited control systems, where switching means that the control is put on or off. These deterministic systems have been studied in a number of papers, with many results in special situations, cf. [39, 49]. In particular, it is known that here, contrary to the situation for autonomous linear control systems, controllability does not imply stabilizability with arbitrary decay rates, as recalled in Proposition 1.23.

The analysis of random switched systems in the present chapter is based on the classical Multiplicative Ergodic Theorem due to Oseledets (see, e.g., [13]). It turns out that a direct application of this theorem to systems in continuous time with random switching is not feasible, since in general they do not define random dynamical systems in the sense of [13] (cf. Example 2.6). Instead, we apply the Multiplicative Ergodic Theorem to an associated system in discrete time and then deduce results for the Lyapunov exponents of the continuous-time system. We remark that Lyapunov exponents for continuous-time systems with random switching are also considered in [112], where one assumes from the beginning that one has random dynamical systems, using hence the classical Multiplicative Ergodic Theorem.

The considered linear equations with random switching form Piecewise Deterministic Markov Processes (PDMP). These processes were introduced by Davis in [65] and have since been extensively studied in the literature. For further references and an analysis of their invariant measures, and in particular their supports, see, for instance, [17,29]. An important particular case which also attracts much research interest is that of Markov Jump Linear

Systems (MJLS), in which one assumes that the random switching signal is generated by a continuous-time Markov chain. For more details of such systems, we refer to [31, 41, 58, 59, 68]. The case of nonlinear switched systems with random switching signals has also been considered in the literature, such as in [40], where multiple Lyapunov functions are used to derive a stability criterion under some slow switching condition that contains as a particular case switching signals coming from continuous-time Markov chains. We also remark that several different notions of stability for systems with random switching have been used in the literature; see, e.g., [69] for a comparison between the usual notions in the context of MJLS. The one considered in this chapter is that of almost sure stability.

The main results of this chapter are (a) a Multiplicative Ergodic Theorem, Theorem 2.31, for linear continuous-time switching systems. This is based on a careful analysis of the relations between the Lyapunov exponents for an associated discrete-time system — which does define a random dynamical system — and those for the system in continuous time; and (b) Theorem 2.36 showing that arbitrary decay rates may be achieved for linear control systems with random switching by choosing appropriate linear feedback laws. This is in contrast to the situation for deterministic switching by persistent excitations, as mentioned above.

The contents of this chapter is as follows. Section 2.2 constructs the random signals acting on the coefficients of the continuous-time system. Example 2.6 shows that, in general, one does not obtain a random dynamical system and Remark 2.7 discusses the relation to previous works in the literature. Section 2.3 introduces an associated system in discrete time, shows that it defines a random dynamical system, and discusses the relations between the Lyapunov exponents for continuous and discrete time. This leads to the formulation of a Multiplicative Ergodic Theorem for the continuous-time system in Section 2.4. Section 2.5 derives a formula for the maximal Lyapunov exponent. Finally, Section 2.6 presents the application to almost sure stabilization with arbitrary decay rate of linear control systems with random switching signals.

2.2 Continuous-time linear switched system and random switching signals

Let $N, d \in \mathbb{N}^*$ and $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{R})$. This chapter considers the continuous-time linear switched system

$$\dot{x}(t) = A_{\alpha(t)} x(t), \tag{2.1}$$

where the switching signal α belongs to the set $\mathcal{P}$ defined by

$$\mathcal{P} = \{\alpha : \mathbb{R}_+ \to \underline{N} \text{ piecewise constant and right continuous}\}.$$

Recall that a piecewise constant function has only finitely many discontinuity points on any bounded interval. Given an initial condition $x_0 \in \mathbb{R}^d$ and $\alpha \in \mathcal{P}$, (2.1) admits a unique solution defined on $\mathbb{R}_+$, which we denote by $\varphi_c(\cdot; x_0, \alpha)$. In order to simplify the notation, for $i \in \underline{N}$, we denote by Φ^i the linear flow defined by the matrix A_i, i.e., $\Phi_t^i = e^{A_i t}$ for every $t \in \mathbb{R}$.

We consider in this chapter that the signal α is randomly generated according to a Markov process which we describe now. Let $M \in \mathcal{M}_N(\mathbb{R})$ be a stochastic matrix, i.e., M has nonnegative entries and $\sum_{j=1}^N M_{ij} = 1$ for every $i \in \underline{N}$. Let p be a probability vector in $\mathbb{R}^N$, i.e., $p \in [0,1]^N$ and $\sum_{i=1}^N p_i = 1$. When necessary, we will regard p as a row vector $p = (p_1, \dots, p_N) \in \mathcal{M}_{1,N}(\mathbb{R})$. We assume in this chapter that p is invariant under M, i.e., that $pM = p$. Finally, let $\mu_1, \dots, \mu_N$ be probability measures on $\mathbb{R}_+^*$ with the Borel σ-algebra $\mathfrak{B}$ and with finite expectation, i.e., $\int_{\mathbb{R}_+^*} t d\mu_i(t) < \infty$ for every $i \in \underline{N}$. Whenever necessary, we will use that $\mu_1, \dots, \mu_N$ define probability measures on $\mathbb{R}_+$ with its Borel σ-algebra, that we also denote by $\mathfrak{B}$ for simplicity.

The random model for the signal α can be described as follows. We choose an initial state $i \in \underline{N}$ according to the probability law defined by p. Then, at every time the system switches to a state i, we choose a random positive time T according to the probability law μ_i and stay in i during the time T, before switching to the next state, which is chosen randomly according to the probability law corresponding to the i-th row $(M_{ij})_{j=1}^N$ of the matrix M. Let us perform this construction more precisely. Recall the construction of product σ-algebras (see, e.g., [89, §§38 and 49]).

Definition 2.1. Let $\Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*}$ and provide Ω with the product σ-algebra $\mathfrak{F} = (\mathfrak{P}(\underline{N}) \times \mathfrak{B})^{\mathbb{N}^*}$. Endow $(\Omega, \mathfrak{F})$ with the probability measure $\mathbb{P}$ defined, for $n \in \mathbb{N}^*$, $i_1, \dots, i_n \in \underline{N}$, and $U_1, \dots, U_n \in \mathfrak{B}$ by

$$\begin{aligned}&\mathbb{P}\Big(\{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}\Big)\\ &= p_{i_1} \mu_{i_1}(U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n).\end{aligned}$$

For a given measurable space X, we denote by $\operatorname{Pr}(X)$ the set of all probability measures on X. The next result shows that the construction from Definition 2.1 is actually a Markov chain in the state space $\underline{N} \times \mathbb{R}_+$. For the definitions of Markov process and its transition probability, initial law, and transition operator, we refer to [80].

Proposition 2.2. *For $n \in \mathbb{N}^*$, let $x_n : \Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*} \to \underline{N} \times \mathbb{R}_+$ denote the canonical projection onto the n-th coordinate. Then $(x_n)_{n=1}^\infty$ is the unique Markov process in $\underline{N} \times \mathbb{R}_+$ with transition probability $P : \underline{N} \times \mathbb{R}_+ \to \operatorname{Pr}(\underline{N} \times \mathbb{R}_+)$ defined by*

$$P(i,t)(\{j\} \times U) = M_{ij}\mu_j(U), \qquad \forall i, j \in \underline{N},\ \forall t \in \mathbb{R}_+,\ \forall U \in \mathfrak{B}, \tag{2.2}$$

and with initial law ν_1 given by

$$\nu_1(\{j\} \times U) = p_j \mu_j(U), \qquad \forall j \in \underline{N},\ \forall U \in \mathfrak{B}. \tag{2.3}$$

The transition operator $T : \operatorname{Pr}(\underline{N} \times \mathbb{R}_+) \to \operatorname{Pr}(\underline{N} \times \mathbb{R}_+)$ of this chain is given by

$$T\nu(\{j\} \times U) = \sum_{i=1}^N \nu(\{i\} \times \mathbb{R}_+) M_{ij} \mu_j(U), \qquad \forall j \in \underline{N},\ \forall U \in \mathfrak{B}. \tag{2.4}$$

Proof. Observe that $\underline{N} \times \mathbb{R}_+$ is a complete separable metric space. Then by [80, Proposition 2.38], it suffices to show that, for every $n \in \mathbb{N}^*$, $i_1, \dots i_n \in \underline{N}$, and $U_1, \dots, U_n \in \mathfrak{B}$,

$$\begin{aligned}&\mathbb{P}\Big(\{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}\Big)\\ &= \int_{\{i_1\}\times U_1} \int_{\{i_2\}\times U_2} \cdots \int_{\{i_{n-1}\}\times U_{n-1}} P(i_{n-1}, t_{n-1})(\{i_n\} \times U_n)\\ &\qquad dP(i_{n-2}, t_{n-2})(i_{n-1}, t_{n-1}) \cdots dP(i_1, t_1)(i_2, t_2) d\nu_1(i_1, t_1).\end{aligned} \tag{2.5}$$

The definitions (2.2) and (2.3) of P and ν_1 immediately give

$$\begin{aligned}&\int_{\{i_1\}\times U_1} \int_{\{i_2\}\times U_2} \cdots \int_{\{i_{n-1}\}\times U_{n-1}} P(i_{n-1}, t_{n-1})(\{i_n\} \times U_n)\\ &\qquad dP(i_{n-2}, t_{n-2})(i_{n-1}, t_{n-1}) \cdots dP(i_1, t_1)(i_2, t_2) d\nu_1(i_1, t_1)\end{aligned}$$

$$
\begin{aligned}
&= \int_{U_1}\int_{U_2}\cdots\int_{U_{n-1}} M_{i_{n-1}i_n}\mu_{i_n}(U_n)M_{i_{n-2}i_{n-1}}d\mu_{i_{n-1}}(t_{n-1})\cdots M_{i_1 i_2}d\mu_{i_2}(t_2)p_{i_1}d\mu_{i_1}(t_1)\\
&= M_{i_{n-1}i_n}\mu_{i_n}(U_n)M_{i_{n-2}i_{n-1}}\mu_{i_{n-1}}(U_{n-1})\cdots M_{i_1 i_2}\mu_{i_2}(U_2)p_{i_1}\mu_{i_1}(U_1),
\end{aligned}
$$

and thus (2.5) holds. The expression of the transition operator follows immediately from its definition (see, e.g., [80, Definition 2.31]). ■

Remark 2.3. The canonical projection of $\underline{N}\times\mathbb{R}_+$ onto $\underline{N}$ transforms the Markov chain from Proposition 2.2 into a discrete Markov chain in the finite state space $\underline{N}$ with transition matrix M and initial distribution p.

To construct a random switching signal α from a certain $\omega=(i_n,t_n)_{n=1}^\infty\in\Omega$, we regard $(i_n)_{n=1}^\infty$ as the sequence of states taken by α and t_n as the time spent in the state i_n. For this construction to be well-defined, one needs to check that the switching times of such α tend to ∞. The next proposition shows that this is the case in a subset of Ω of full measure.

Proposition 2.4. *The subset Ω_0 of Ω defined by*

$$
\Omega_0=\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;\sum_{n=1}^\infty t_n=\infty \text{ and } t_n>0 \text{ for every } n\in\mathbb{N}^*\right\}
$$

satisfies $\mathbb{P}(\Omega_0)=1$.

Proof. We write $\Omega_0=\Omega'\cap\Omega''$, with

$$
\begin{aligned}
\Omega'&=\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;\sum_{n=1}^\infty t_n=\infty\right\},\\
\Omega''&=\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;t_n>0 \text{ for every } n\in\mathbb{N}^*\right\}.
\end{aligned}
$$

Then it follows that

$$
\mathbb{P}(\Omega'')=\mathbb{P}\left(\bigcap_{n=1}^\infty\{(i_j,t_j)_{j=1}^\infty\in\Omega\mid t_n>0\}\right)=1,
$$

since for every n

$$
\mathbb{P}\{(i_j,t_j)_{j=1}^\infty\in\Omega\mid t_n>0\}=\sum_{(i_1,\dots,i_n)\in\underline{N}^n}p_{i_1}M_{i_1i_2}\cdots M_{i_{n-1}i_n}\mu_{i_n}((0,\infty))=1.
$$

Denoting by Ω'^c the complement of Ω' in Ω, we have

$$
\begin{aligned}
\Omega'^c&=\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;\sum_{n=1}^\infty t_n<\infty\right\}\subset\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;\lim_{n\to\infty}t_n=0\right\}\\
&=\bigcap_{k=1}^\infty\bigcup_{r=1}^\infty\bigcap_{m=r}^\infty\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;t_m<\frac{1}{k}\right\}.
\end{aligned}\tag{2.6}
$$

For $k,r,K\in\mathbb{N}^*$ with $K\geq r$, let

$$
E_K^{r,k}=\bigcap_{m=r}^K\left\{(i_n,t_n)_{n=1}^\infty\in\Omega\;\middle|\;t_m<\frac{1}{k}\right\}=(\underline{N}\times\mathbb{R}_+)^{r-1}\times(\underline{N}\times[0,1/k))^{K-r+1}\times(\underline{N}\times\mathbb{R}_+)^{\mathbb{N}^*\setminus\underline{K}}
$$

$$= \bigcup_{(i_1,\dots,i_K)\in\underline{N}^K} \prod_{j=1}^{r-1}\Big(\{i_j\}\times\mathbb{R}_+\Big)\times\prod_{j=r}^{K}\Big(\{i_j\}\times[0,1/k)\Big)\times(\underline{N}\times\mathbb{R}_+)^{\mathbb{N}^*\setminus\underline{K}}.$$

This union is disjoint, and thus

$$\mathbb{P}\left(E_K^{r,k}\right) = \sum_{(i_1,\dots,i_K)\in\underline{N}^K} p_{i_1}\prod_{j=2}^{K} M_{i_{j-1}i_j}\prod_{j=r}^{K}\mu_{i_j}([0,1/k)) \leq \mu_{\max}(k)^{K-r+1},$$

where $\mu_{\max}(k) = \max_{i\in\underline{N}}\mu_i([0,1/k))$. Then $\mu_{\max}(k)\to 0$ as $k\to\infty$, and hence there exists $k_*\in\mathbb{N}^*$ such that $\mu_{\max}(k_*) < 1$. Since, for every $r,k\in\mathbb{N}^*$, the sequence of sets $\left(E_K^{r,k}\right)_{K=r}^{\infty}$ is decreasing, we obtain that

$$\mathbb{P}\left(\bigcap_{m=r}^{\infty}\left\{(i_n,t_n)_{n=1}^{\infty}\in\Omega \,\middle|\, t_m < \frac{1}{k_*}\right\}\right) = \lim_{K\to\infty}\mathbb{P}\left(E_K^{r,k_*}\right) = 0.$$

This shows that $\mathbb{P}(\Omega') = 1$ thanks to (2.6). ■

We now associate to each $\omega\in\Omega_0$ a signal $\alpha\in\mathcal{P}$.

Definition 2.5. We define the map $\boldsymbol{\alpha}:\Omega_0\to\mathcal{P}$ as follows: for $\omega = (i_n,t_n)_{n=1}^{\infty}\in\Omega_0$, we set $s_0 = 0$, $s_n = \sum_{k=1}^{n} t_k$ for $n\in\mathbb{N}^*$, and $\boldsymbol{\alpha}(\omega)(t) = i_n$ for every $n\in\mathbb{N}^*$ and $t\in[s_{n-1},s_n)$.

Notice that $\boldsymbol{\alpha}$ is well-defined since $\sum_{n=1}^{\infty} t_n = \infty$ for every $\omega = (i_n,t_n)_{n=1}^{\infty}\in\Omega_0$. When necessary, we regard $\boldsymbol{\alpha}$ as a function $\boldsymbol{\alpha}:\Omega\to\mathcal{P}$ defined almost everywhere.

In order to consider solutions of (2.1) for signals α chosen randomly according to the previous construction, we use the solution map φ_{c} of (2.1) to define the map

$$\varphi_{\mathrm{rc}}:\left\{\begin{array}{ccc}\mathbb{R}_+\times\mathbb{R}^d\times\Omega_0 & \to & \mathbb{R}^d\\ (t;x_0,\omega) & \mapsto & \varphi_{\mathrm{c}}(t;x_0,\boldsymbol{\alpha}(\omega)).\end{array}\right. \tag{2.7}$$

A natural idea to study the exponential behavior of the switched system with random switching signals described by φ_{rc} would be to apply the continuous-time Oseledets' Multiplicative Ergodic Theorem (see, e.g., [13, Theorem 3.4.1]) to obtain information on the Lyapunov exponents for φ_{rc}. To do so, φ_{rc} should define a random dynamical system on $\mathbb{R}^d\times\Omega$, i.e., one would have to provide a metric dynamical system θ on Ω — a measurable dynamical system $\theta:\mathbb{R}_+\times\Omega\to\Omega$ on $(\Omega,\mathcal{F},\mathbb{P})$ such that θ_t preserves $\mathbb{P}$ for every $t\geq 0$ — in such a way that φ_{rc} becomes a cocycle over θ (for the precise definitions of random dynamical system, metric dynamical system, and cocycle, see, e.g., [13]). However, in general the natural choice for θ to obtain the cocycle property for φ_{rc}, namely the time shift, does not define such a measure preserving map, as shown in the following example.

Example 2.6. For $t\geq 0$, let $\theta_t:\Omega\to\Omega$ be defined for almost every $\omega\in\Omega$ as follows. For $\omega = (i_j,t_j)_{j=1}^{\infty}\in\Omega_0$, set $s_0 = 0$, $s_k = \sum_{j=1}^{k} t_j$ for $k\in\mathbb{N}^*$. Let $n\in\mathbb{N}^*$ be the unique integer such that $t\in[s_{n-1},s_n)$. We define $\theta_t(\omega) = (i_j^*,t_j^*)_{j=1}^{\infty}$ by $i_j^* = i_{n+j-1}$ for $j\in\mathbb{N}^*$, $t_1^* = s_n - t$, $t_j^* = t_{n+j-1}$ for $j\geq 2$. One immediately verifies that θ_t corresponds to the time shift in $\mathcal{P}$, i.e., that, for every $t,s\geq 0$ and $\omega\in\Omega_0$, one has

$$\boldsymbol{\alpha}(\theta_t\omega)(s) = \boldsymbol{\alpha}(\omega)(t+s).$$

However, the map θ_t in $(\Omega,\mathcal{F})$ does not preserve the measure $\mathbb{P}$ in general. Indeed, suppose that $\mu_i = \delta_1$ for every $i\in\underline{N}$, where δ_1 denotes the Dirac measure at 1. In particular, a set

$E \in \mathfrak{F}$ has nonzero measure only if E contains a point $(i_j, t_j)_{j=1}^{\infty}$ with $t_j = 1$ for every $j \in \mathbb{N}^*$. For $r \in \mathbb{N}^*$ and $i_1, \dots, i_r \in \underline{N}$, let

$$E = \{i_1\} \times \{1\} \times \cdots \times \{i_r\} \times \{1\} \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{r}}.$$

Then $\mathbb{P}(E) = p_{i_1} M_{i_1 i_2} \cdots M_{i_{r-1} i_r}$, and, for $t \geq 0$, $\theta_t^{-1}(E)$ is the set of points $(i_j^*, t_j^*)_{j=1}^{\infty}$ such that, setting $s_0^* = 0$, $s_k^* = \sum_{j=1}^{k} t_j^*$ for $k \in \mathbb{N}^*$, and $n \in \mathbb{N}^*$ the unique integer such that $t \in \left[s_{n-1}^*, s_n^*\right)$, one has $s_n^* - t = 1$, $t_{n+j-1}^* = 1$ for $j = 2, \dots, r$, and $i_{n+j-1}^* = i_j$ for $j \in \underline{r}$. If $t \notin \mathbb{N}$, then $s_n^* = t + 1 \notin \mathbb{N}$, and thus there exists $j \in \underline{n}$ such that $t_j^* \neq 1$. We have shown that, if $t \notin \mathbb{N}$, then, for every $\omega = (i_j^*, t_j^*)_{j=1}^{\infty} \in \theta_t^{-1}(E)$, there exists $j \in \mathbb{N}^*$ such that $t_j^* \neq 1$, and thus $\mathbb{P}(\theta_t^{-1}(E)) = 0$, hence θ_t does not preserve the measure $\mathbb{P}$.

Remark 2.7. For some particular choices of $\mu_1, \dots, \mu_N$, the time-shift θ_t may preserve $\mathbb{P}$, in which case the continuous-time Multiplicative Ergodic Theorem can be applied directly to (2.7). This special case falls in the framework of [112]. An important particular case where θ_t preserves $\mathbb{P}$ is when $\mu_1, \dots, \mu_N$ are chosen in such a way that $\boldsymbol{\alpha}$ becomes a homogeneous continuous-time Markov chain, which is the case treated, e.g., in [31, 68]. Our stability results from Section 2.5 generalize the corresponding almost sure stability criteria from [31, 68, 112].

2.3 Associated discrete-time system and Lyapunov exponents

Example 2.6 shows that in general one cannot expect to obtain a random dynamical system from φ_{rc} in order to apply the continuous-time Oseledets' Multiplicative Ergodic Theorem. Our strategy to study the exponential behavior of φ_{rc} relies instead on defining a suitable discrete-time map φ_{rd} associated with φ_{rc}, in such a way that φ_{rd} does define a discrete-time random dynamical system — to which the discrete-time Oseledets' Multiplicative Ergodic Theorem can be applied (see, e.g., [13, Theorem 3.4.1]) — and that the exponential behavior of φ_{rc} and φ_{rd} can be compared.

2.3.1 Associated discrete-time deterministic system

In this subsection we define a discrete-time deterministic system from the continuous-time system (2.1) determined by its solution map φ_{c}.

Definition 2.8. We say that an increasing sequence $\sigma = (s_n)_{n=0}^{\infty}$ of nonnegative real numbers with $s_0 = 0$ and $\lim_{n\to\infty} s_n = \infty$ is *compatible* with a signal $\alpha \in \mathcal{P}$ if $\alpha|_{[s_n, s_{n+1})}$ is constant for every $n \in \mathbb{N}$, and we denote

$$\mathcal{Q} = \{(\alpha, \sigma) \in \mathcal{P} \times \mathbb{R}_+^{\mathbb{N}} \mid \sigma \text{ is compatible with } \alpha\}.$$

For $(\alpha, \sigma) \in \mathcal{Q}$ with $\sigma = (s_n)_{n=0}^{\infty}$, we consider the difference equation

$$x_{n+1} = e^{A_{\alpha(s_n)}(s_{n+1} - s_n)} x_n. \tag{2.8}$$

System (2.8) is obtained from (2.1) by taking the values of a continuous-time solution at the discrete times s_n. The sequence $(s_n)_{n=0}^{\infty}$ contains all the discontinuities of α and may also contain times with trivial jumps. The solution of (2.8) associated with $(\alpha, \sigma) \in \mathcal{Q}$ and with initial condition $x_0 \in \mathbb{R}^d$ is denoted by $\varphi_{\mathrm{d}}(\cdot; x_0, \alpha, \sigma)$. Notice that the solution maps φ_{c} and φ_{d} satisfy, for every $x_0 \in \mathbb{R}^d$ and $(\alpha, \sigma) \in \Omega$,

$$\begin{aligned} &\varphi_{\mathrm{c}}(0; x_0, \alpha) = x_0, \\ &\varphi_{\mathrm{c}}(t; x_0, \alpha) = \Phi_{t-s_n}^{\alpha(s_n)}(\varphi_{\mathrm{c}}(s_n; x_0, \alpha)), \text{ if } t \in (s_n, s_{n+1}] \text{ for some } n \in \mathbb{N}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \varphi_{\mathrm{d}}(0; x_0, \alpha, \sigma) &= x_0, \\ \varphi_{\mathrm{d}}(n+1; x_0, \alpha, \sigma) &= \Phi_{s_{n+1}-s_n}^{\alpha(s_n)}(\varphi_{\mathrm{d}}(n; x_0, \alpha, \sigma)), \text{ for } n \in \mathbb{N}. \end{aligned} \tag{2.10}$$

It follows immediately that, for every $n \in \mathbb{N}$,

$$\varphi_{\mathrm{c}}(s_n; x_0, \alpha) = \varphi_{\mathrm{d}}(n; x_0, \alpha, \sigma). \tag{2.11}$$

We characterize the asymptotic behavior of systems (2.1) and (2.8) by considering the associated Lyapunov exponents defined as follows.

Definition 2.9. Let $(\alpha, \sigma) \in \Omega$ and $x_0 \in \mathbb{R}^d \setminus \{0\}$. The *Lyapunov exponent* for the continuous-time system (2.1) is

$$\lambda_{\mathrm{c}}(x_0, \alpha) = \limsup_{t \to \infty} \frac{1}{t} \log \left|\varphi_{\mathrm{c}}(t; x_0, \alpha)\right| \tag{2.12}$$

and the *Lyapunov exponent* for the discrete-time system (2.8) is

$$\lambda_{\mathrm{d}}(x_0, \alpha, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \log \left|\varphi_{\mathrm{d}}(n; x_0, \alpha, \sigma)\right|. \tag{2.13}$$

The main difference between (2.12) and (2.13) lies in the terms $\frac{1}{t}$ and $\frac{1}{n}$. In order to be able to compare them asymptotically, one needs an additional hypothesis.

Definition 2.10. Let $(\alpha, \sigma) \in \Omega$ with $\sigma = (s_n)_{n=0}^{\infty}$. We say that (α, σ) is *regular* if the limit

$$m(\alpha, \sigma) = \lim_{n \to \infty} \frac{s_n}{n} \tag{2.14}$$

exists and is a positive real number.

Theorem 2.11. *Suppose that $(\alpha, \sigma) \in \Omega$ is regular. Then, for every $x_0 \in \mathbb{R}^d \setminus \{0\}$, the Lyapunov exponents of the continuous- and discrete-time systems (2.1) and (2.8) are related by*

$$\lambda_{\mathrm{d}}(x_0, \alpha, \sigma) = m(\alpha, \sigma) \lambda_{\mathrm{c}}(x_0, \alpha).$$

Proof. Write $\sigma = (s_n)_{n=0}^{\infty}$. Let us first show that $\lambda_{\mathrm{d}}(x_0, \alpha, \sigma) \leq m(\alpha, \sigma) \lambda_{\mathrm{c}}(x_0, \alpha)$. For every $n \in \mathbb{N}^*$, one has, by (2.11),

$$\frac{1}{n} \log \left|\varphi_{\mathrm{d}}(n; x_0, \alpha, \sigma)\right| = \frac{s_n}{n} \frac{1}{s_n} \log \left|\varphi_{\mathrm{c}}(s_n; x_0, \alpha)\right|.$$

One clearly has $\limsup_{n \to \infty} \frac{1}{s_n} \log \left|\varphi_{\mathrm{c}}(s_n; x_0, \alpha)\right| \leq \limsup_{t \to \infty} \frac{1}{t} \log \left|\varphi_{\mathrm{c}}(t; x_0, \alpha)\right|$ and then the conclusion follows since $\frac{s_n}{n} \to m(\alpha, \sigma)$.

We now turn to the proof of the inequality $\lambda_{\mathrm{d}}(x_0,\alpha,\sigma) \geq m(\alpha,\sigma)\lambda_{\mathrm{c}}(x_0,\alpha)$. Let $C, \gamma > 0$ be such that $\left|\Phi_t^i x\right| \leq Ce^{\gamma t}|x|$ for every $i \in \underline{N}$, $x \in \mathbb{R}^d$, and $t \geq 0$. For $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$, let $n_t \in \mathbb{N}$ be the unique integer such that $t \in \left(s_{n_t}, s_{n_t+1}\right]$. Then

$$\begin{aligned}\frac{1}{t}\log|\varphi_{\mathrm{c}}(t;x_0,\alpha)| &= \frac{1}{t}\log\left|\Phi_{t-s_{n_t}}^{\alpha(s_{n_t})}(\varphi_{\mathrm{c}}(s_{n_t};x_0,\alpha))\right| = \frac{1}{t}\log\left|\Phi_{t-s_{n_t}}^{\alpha(s_{n_t})}(\varphi_{\mathrm{d}}(n_t;x_0,\alpha,\sigma))\right| \\ &\leq \frac{\log C}{t} + \gamma\frac{t-s_{n_t}}{t} + \frac{1}{t}\log|\varphi_{\mathrm{d}}(n_t;x_0,\alpha,\sigma)|.\end{aligned} \tag{2.15}$$

Since $t \in \left(s_{n_t}, s_{n_t+1}\right]$, one has

$$0 \leq \frac{t-s_{n_t}}{t} \leq \frac{s_{n_t+1}}{s_{n_t}} - 1 \xrightarrow[t\to\infty]{} 0, \tag{2.16}$$

where we use (2.14) to obtain that $\frac{s_{n_t+1}}{s_{n_t}} \to 1$ as $t \to \infty$. We write $\frac{1}{t} = \frac{n_t}{t}\frac{1}{n_t}$. Since $t \in \left(s_{n_t}, s_{n_t+1}\right]$, one has $\frac{n_t}{t} \in \left[\frac{n_t}{s_{n_t+1}}, \frac{n_t}{s_{n_t}}\right)$. Now

$$\lim_{t\to\infty}\frac{n_t}{s_{n_t}} = \frac{1}{m(\alpha,\sigma)} \qquad \text{and} \qquad \lim_{t\to\infty}\frac{n_t}{s_{n_t+1}} = \lim_{t\to\infty}\left(\frac{n_t+1}{s_{n_t+1}} - \frac{1}{s_{n_t+1}}\right) = \frac{1}{m(\alpha,\sigma)},$$

and thus $\frac{n_t}{t} \to \frac{1}{m(\alpha,\sigma)}$ as $t \to \infty$. Using this fact and inserting (2.16) into (2.15), one obtains the conclusion of the theorem by letting $t \to \infty$. ■

2.3.2 Discrete-time random dynamical system

We have constructed, in Section 2.3.1, the discrete-time system (2.8) associated with the continuous-time system (2.1). In this subsection, we use (2.8) and the probabilistic setting from Section 2.2 to construct a random dynamical system in discrete time, to which we will apply Oseledets' Multiplicative Ergodic Theorem in Section 2.4. Thanks to Theorem 2.11, this will allow us also to get information on the Lyapunov exponents of the continuous-time system. In order to perform this construction, one needs to choose, for each $\omega \in \Omega_0$, a sequence σ compatible with $\boldsymbol{\alpha}(\omega)$.

A sequence σ that is compatible with a certain $\alpha \in \mathcal{P}$ corresponds to a sequence of times where we observe the continuous-time solution map φ_{c} to define the discrete-time map φ_{d}. A natural choice, considering the fact that the probabilistic model from Definition 2.1 is a Markov chain, is to choose σ as the sequence of transition times of this chain, as follows.

Definition 2.12. We define the map $\mathbf{s} : \Omega_0 \to \mathbb{R}_+^{\mathbb{N}}$ as follows: for $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$, we set $\mathbf{s}(\omega) = (s_n(\omega))_{n=0}^{\infty}$ with $s_0(\omega) = 0$, $s_n(\omega) = \sum_{k=1}^{n} t_k$ for $n \in \mathbb{N}^*$.

Notice that, for every $\omega \in \Omega_0$, $\mathbf{s}(\omega)$ is compatible with $\boldsymbol{\alpha}(\omega)$. We define the random discrete-time system φ_{rd} by

$$\varphi_{\mathrm{rd}} : \left\{\begin{array}{ccc} \mathbb{N}\times\mathbb{R}^d\times\Omega_0 & \to & \mathbb{R}^d \\ (n;x_0,\omega) & \mapsto & \varphi_{\mathrm{d}}(n;x_0,\boldsymbol{\alpha}(\omega),\mathbf{s}(\omega)).\end{array}\right. \tag{2.17}$$

We also define the random Lyapunov exponents λ_{rc} and λ_{rd} for $x_0 \in \mathbb{R}^d \setminus \{0\}$ and almost every $\omega \in \Omega$ by

$$\lambda_{\mathrm{rc}}(x_0,\omega) = \lambda_{\mathrm{c}}(x_0,\boldsymbol{\alpha}(\omega)), \qquad \lambda_{\mathrm{rd}}(x_0,\omega) = \lambda_{\mathrm{d}}(x_0,\boldsymbol{\alpha}(\omega),\mathbf{s}(\omega)). \tag{2.18}$$

A natural way to define a discrete-time metric dynamical system on $(\Omega, \mathfrak{F}, \mathbb{P})$ is to consider the shift operator. Let $\theta : \Omega \to \Omega$ be defined by

$$\theta((i_n, t_n)_{n=1}^{\infty}) = (i_{n+1}, t_{n+1})_{n=1}^{\infty}. \tag{2.19}$$

Proposition 2.13. *The measure $\mathbb{P}$ is invariant under θ.*

Proof. It suffices to show that $\mathbb{P}(\theta^{-1}(E)) = \mathbb{P}(E)$ for every set E of the form

$$E = \{i_1\} \times U_1 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}$$

for some $n \in \mathbb{N}^*$, $i_1, \dots, i_n \in \underline{N}$, and $U_1, \dots, U_n \in \mathfrak{B}$. For such a set E, we have

$$\theta^{-1}(E) = \bigcup_{i=1}^{N} \{i\} \times \mathbb{R}_+ \times \{i_1\} \times U_1 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n+1}},$$

and the previous union is disjoint, so that

$$\begin{aligned}\mathbb{P}(\theta^{-1}(E)) &= \sum_{i=1}^{N} p_i \mu_i(\mathbb{R}_+) M_{i i_1} \mu_{i_1}(U_1) \prod_{j=2}^{n} M_{i_{j-1} i_j} \mu_{i_j}(U_j) = \left(\sum_{i=1}^{N} p_i M_{i i_1} \right) \mu_{i_1}(U_1) \prod_{j=2}^{n} M_{i_{j-1} i_j} \mu_{i_j}(U_j) \\ &= p_{i_1} \mu_{i_1}(U_1) \prod_{j=2}^{n} M_{i_{j-1} i_j} \mu_{i_j}(U_j) = \mathbb{P}(E),\end{aligned}$$

since $pM = p$. ■

Thanks to Proposition 2.13, θ is a discrete-time metric dynamical system in $(\Omega, \mathfrak{F}, \mathbb{P})$. Moreover, since the shift operator $\theta : \Omega \to \Omega$ satisfies $\theta(\Omega_0) = \Omega_0$, θ also defines a metric dynamical system in $(\Omega_0, \mathfrak{F}, \mathbb{P})$ (where $\mathfrak{F}$ and $\mathbb{P}$ are understood to be restricted to Ω_0).

An important question regarding the metric dynamical system θ in $(\Omega, \mathfrak{F}, \mathbb{P})$ is to determine whether it is ergodic. To characterize the cases where such ergodicity holds, we start by providing the following definition.

Definition 2.14. Let $(\Omega, \mathfrak{F})$ be the measurable space from Definition 2.1 and $\nu \in \Pr(\underline{N} \times \mathbb{R}_+)$. We define the probability measure $\mathbb{P}_\nu$ in $(\Omega, \mathfrak{F})$ by requiring that, for every $n \in \mathbb{N}^*$, $i_1, \dots, i_n \in \underline{N}$, and $U_1, \dots, U_n \in \mathfrak{B}$,

$$\begin{aligned}&\mathbb{P}_\nu\left(\{i_1\} \times U_1 \times \{i_2\} \times U_2 \times \cdots \times \{i_n\} \times U_n \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}\right) \\ &= \nu(\{i_1\} \times U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n).\end{aligned} \tag{2.20}$$

Remark 2.15. If $\nu(\{i\} \times U) = p_i \mu_i(U)$ for every $i \in \underline{N}$ and $U \in \mathfrak{B}$, then $\mathbb{P}_\nu$ coincides with the measure $\mathbb{P}$ from Definition 2.1. Moreover, as in Proposition 2.2, for every $\nu \in \Pr(\underline{N} \times \mathbb{R}_+)$, $\mathbb{P}_\nu$ is the probability measure associated with a Markov process in $\underline{N} \times \mathbb{R}_+$ with transition probability P given by (2.2), transition operator T given by (2.4), and with initial law ν.

Lemma 2.16. *The measure $\mathbb{P}_\nu$ is invariant under the shift θ if and only if $\nu(\{i\} \times U) = \nu(\{i\} \times \mathbb{R}_+)\mu_i(U)$ for every $i \in \underline{N}$, $U \in \mathfrak{B}$, and $(\nu(\{i\} \times \mathbb{R}_+))_{i=1}^{N}$ is a left eigenvector of M associated with the eigenvalue 1.*

Proof. Notice that $\mathbb{P}_\nu$ is invariant under θ if and only if $T\nu = \nu$. Hence $\mathbb{P}_\nu$ is invariant under θ if and only if

$$\nu(\{j\} \times U) = \sum_{i=1}^{N} \nu(\{i\} \times \mathbb{R}_+) M_{ij} \mu_j(U), \qquad \forall j \in \underline{N},\ \forall U \in \mathfrak{B}. \tag{2.21}$$

If (2.21) holds, we apply it to $U = \mathbb{R}_+$ to get that $(\nu(\{i\}\times\mathbb{R}_+))_{i=1}^N$ is a left eigenvector of M associated with the eigenvalue 1, and it then follows that $\nu(\{j\}\times U) = \nu(\{j\}\times\mathbb{R}_+)\mu_j(U)$ for every $j \in \underline{N}$ and $U \in \mathfrak{B}$. The converse is immediate. ■

Let

$$V = \left\{ q \in [0,1]^N \;\middle|\; \sum_{i=1}^N q_i = 1 \text{ and } qM = q \right\}, \tag{2.22}$$

which is a non-empty convex subset of $\mathbb{R}^N$. An element $q \in V$ is said to be *extremal* if it cannot be written as $q = tq_1 + (1-t)q_2$ for some $t \in (0,1)$ and $q_1, q_2 \in V$ with $q_1 \neq q_2$. With each $q \in V$, we associate a probability measure $\nu_q \in \Pr(\underline{N}\times\mathbb{R}_+)$ by setting

$$\nu_q(\{i\}\times U) = q_i\mu_i(U), \qquad \forall i \in \underline{N},\ U \in \mathfrak{B}. \tag{2.23}$$

As a consequence of Lemma 2.16, the map $q \mapsto \nu_q$ is a linear bijection between V and the set of all probability measures ν in $\underline{N}\times\mathbb{R}_+$ for which $\mathbb{P}_\nu$ is invariant under the shift θ. Hence, one obtains immediately from [80, Theorem 5.7] the following result.

Proposition 2.17. *Let $q \in V$. The metric dynamical system θ is ergodic in $\left(\Omega, \mathfrak{F}, \mathbb{P}_{\nu_q}\right)$ if and only if q is an extremal of V.*

Remark 2.18. When M is irreducible, V contains only one point q and hence θ is ergodic for the measure $\mathbb{P}_{\nu_q}$.

Now that we have defined the random discrete-time system (2.17) and provided the metric dynamical system θ, we can show that the pair $(\theta, \varphi_{\mathrm{rd}})$ defines a random dynamical system.

Proposition 2.19. *$(\theta, \varphi_{\mathrm{rd}})$ is a discrete-time random dynamical system over $(\Omega, \mathfrak{F}, \mathbb{P})$.*

Proof. Since θ is a discrete-time metric dynamical system over $(\Omega, \mathfrak{F}, \mathbb{P})$, one is only left to show that φ_{rd} satisfies the cocycle property

$$\varphi_{\mathrm{rd}}(n+m; x_0, \omega) = \varphi_{\mathrm{rd}}(n; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)), \quad \forall n, m \in \mathbb{N},\ \forall x_0 \in \mathbb{R}^d,\ \forall \omega \in \Omega_0. \tag{2.24}$$

Let $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$. Then it follows immediately from the definitions of $\boldsymbol{\alpha}$ and $\mathbf{s}$ that for $n, m \in \mathbb{N}$,

$$\begin{gathered} s_n(\theta^m(\omega)) = \sum_{k=1}^n t_{k+m} = \sum_{k=m+1}^{m+n} t_k = s_{n+m}(\omega) - s_m(\omega), \\ \boldsymbol{\alpha}(\theta^m(\omega))(s_n(\theta^m(\omega))) = i_{n+m} = \boldsymbol{\alpha}(\omega)(s_{n+m}(\omega)). \end{gathered}$$

We prove (2.24) by induction on n. When $n = 0$, (2.24) is clearly satisfied for every $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, and $\omega \in \Omega_0$. Suppose now that $n \in \mathbb{N}$ is such that (2.24) is satisfied for every $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, and $\omega \in \Omega_0$. Using (2.10), we obtain

$$\begin{aligned} \varphi_{\mathrm{rd}}(n+1; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)) &= \Phi^{\boldsymbol{\alpha}(\theta^m(\omega))(s_n(\theta^m(\omega)))}_{s_{n+1}(\theta^m(\omega)) - s_n(\theta^m(\omega))}(\varphi_{\mathrm{rd}}(n; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega))) \\ &= \Phi^{\boldsymbol{\alpha}(\omega)(s_{n+m}(\omega))}_{s_{n+m+1}(\omega) - s_{n+m}(\omega)}(\varphi_{\mathrm{rd}}(n+m; x_0, \omega)) = \varphi_{\mathrm{rd}}(n+m+1; x_0, \omega), \end{aligned}$$

which concludes the proof of (2.24). ■

Since our goal is to compare the asymptotic behavior of (2.7) and (2.17) using Theorem 2.11, we need to show that $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$ is regular for almost every $\omega \in \Omega$. To do so, we first consider the structure of the matrix M, using classical notation for Markov chains (see, e.g., [156]).

Definition 2.20. Let $M \in \mathcal{M}_N(\mathbb{R})$ be a stochastic matrix. For $i, j \in \underline{N}$, we say that i *leads to* j if $i = j$ or there exist $r \in \mathbb{N}^*$ and $i_1, \dots, i_r \in \underline{N}$ such that $M_{ii_1} M_{i_1 i_2} \cdots M_{i_r j} > 0$. We say that i *and* j *communicate* if i leads to j and i leads to j. This is an equivalence relation and we decompose $\underline{N}$ into the corresponding $R' \in \mathbb{N}^*$ equivalence classes $C_1, \dots, C_{R'}$. For $i, j \in \underline{R'}$, we say that C_i *leads to* C_j if there exist $i^* \in C_i$ leading to some $j^* \in C_j$. A class C_i is said to be *essential* if it does not lead to another class, and *inessential* otherwise.

At least one essential class exists. Up to a permutation in the sets of indices $\underline{N}$ and $\underline{R'}$, we can assume that $C_1 = \{1, 2, \dots, n_1\}$, $C_2 = \{n_1 + 1, n_1 + 2, \dots, n_1 + n_2\}$, ..., $C_{R'} = \{n_1 + \dots + n_{R'-1} + 1, \dots, n_1 + \dots + n_{R'}\}$ for some $n_1, \dots, n_{R'} \in \mathbb{N}^*$, and that M can be written as

$$M = \left(\begin{array}{ccccc|c} P_1 & 0 & 0 & \cdots & 0 & \\ 0 & P_2 & 0 & \cdots & 0 & \\ 0 & 0 & P_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & P_R & \\ \hline & & * & & & Q \\ & & & & & \end{array}\right), \tag{2.25}$$

where R is the number of essential classes, P_i is the square matrix corresponding to the essential class C_i for $i \in \underline{R}$, and Q is the square matrix corresponding to all inessential classes. The following proposition recalls some classical properties of stochastic matrices. Its proof can be found in textbooks on the subject, such as [156, §4.2].

Proposition 2.21. *Let M be a stochastic matrix decomposed as* (2.25).

(a) *For $i \in \underline{R}$, $P_i \in \mathcal{M}_{n_i}(\mathbb{R})$ is an irreducible stochastic matrix with a unique invariant probability $p^i \in \mathbb{R}^{n_i}$. We extend p^i to a vector in $\mathbb{R}^N$ by setting to 0 its components not in C_i, and write $p^i = (p^i_j)_{j=1}^N$;*

(b) *Every probability vector $q \in \mathbb{R}^N$ invariant under M can be decomposed as $q = \alpha_1 p^1 + \dots + \alpha_R p^R$ for some $\alpha_1, \dots, \alpha_R \in [0, 1]$ with $\sum_{i=1}^R \alpha_i = 1$.*

Remark 2.22. If follows from this proposition that the set V defined in (2.22) is the convex hull of $\{p^1, \dots, p^R\}$, and that $q \in V$ is an extremal of V if and only if $q = p^i$ for some $i \in \underline{R}$.

For a probability vector $q \in [0, 1]^N$, we define the probability measure $\mathbb{P}^q$ in the measurable space $(\Omega, \mathfrak{F})$ by setting $\mathbb{P}^q = \mathbb{P}_{\nu_q}$, where ν_q is defined in (2.23) and $\mathbb{P}_{\nu_q}$ is given in Definition 2.14. Thanks to Lemma 2.16, $\mathbb{P}^q$ is invariant under θ if and only if $qM = q$. Let $\alpha_1 q_1 + \dots + \alpha_k q_k$ be a convex combination of probability vectors $q_1, \dots, q_k \in [0, 1]^N$. Thanks to Definition 2.14 and (2.23), one obtains that, for every $E \in \mathfrak{F}$,

$$\mathbb{P}^{\alpha_1 q_1 + \dots + \alpha_k q_k}(E) = \alpha_1 \mathbb{P}^{q_1}(E) + \dots + \alpha_k \mathbb{P}^{q_k}(E). \tag{2.26}$$

As a consequence of Proposition 2.17 and Remark 2.22, one immediately obtains the following result.

Corollary 2.23. *Let $q \in \mathbb{R}^N$ be a probability vector with $qM = q$ and $p^1, \dots, p^R$ be as in Proposition 2.21(a). The map θ is ergodic for the measure $\mathbb{P}^q$ if and only if $q = p^i$ for some $i \in \underline{R}$.*

We now provide a decomposition of the space Ω according to the essential classes $C_1, \dots, C_R$. For $i \in \underline{R}$, we set

$$\Omega^i = \{\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega \mid \exists n_0 \in \mathbb{N}^* \text{ such that } i_n \in C_i \text{ for } n \geq n_0\},$$

and

$$\Omega^0 = \Omega \setminus \bigcup_{i=1}^{R} \Omega^i.$$

Then clearly $\Omega = \bigcup_{i=0}^{R} \Omega^i$ and the union is disjoint. For $i \in \underline{R}$, the set Ω^i is the set of all sequences $(i_n, t_n)_{n=0}^{\infty}$ such that $(i_n)_{n=0}^{\infty}$ eventually enters the class C_i and remains in this class.

Proposition 2.24. *Let $q = \alpha_1 p^1 + \dots + \alpha_R p^R \in V$ be as in Proposition 2.21(b). Then, for every $i \in \underline{R}$, $\mathbb{P}^q(\Omega^i) = \alpha_i$. In particular, $\mathbb{P}^q(\Omega^0) = 0$.*

Proof. Since the components of p^j corresponding to indices not in C_j are all zero and M has the form (2.25), one obtains that $\mathbb{P}^{p^j}\left(\left(C_j \times \mathbb{R}_+\right)^{\mathbb{N}^*}\right) = 1$, and thus

$$\mathbb{P}^{p^j}(\Omega^i) = \mathbb{P}^{p^j}\left(\Omega^i \cap \left(C_j \times \mathbb{R}_+\right)^{\mathbb{N}^*}\right) = \delta_{ij}, \tag{2.27}$$

where δ_{ij} denotes the Kronecker delta. The conclusion follows immediately from (2.26) and (2.27). ∎

We can now prove that $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$ is regular for almost every $\omega \in \Omega$.

Proposition 2.25. *The map $\omega \mapsto m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$ is invariant under θ and, for every $i \in \underline{R}$ such that $\mathbb{P}(\Omega^i) \neq 0$ and almost every $\omega \in \Omega^i$,*

$$m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega)) = \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t \, d\mu_j(t), \tag{2.28}$$

where, for $i \in \underline{R}$, $p^i = \left(p_j^i\right)_{j=1}^N$ are the probability vectors from Proposition 2.21(a). In particular, $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$ is regular for almost every $\omega \in \Omega$.

Proof. Consider the map $f : \Omega_0 \to \mathbb{R}_+^*$ given by $f((i_n, t_n)_{i=1}^{\infty}) = t_1$. For every $k \in \mathbb{N}$, $f \circ \theta^k((i_n, t_n)_{n=1}^{\infty}) = t_{k+1}$. By Birkhoff's Ergodic Theorem (see, e.g., [146, Chapter 2, Theorem 2.3]), there exists a function $f^* \in L^1(\Omega, \mathbb{R}_+)$, invariant under θ, such that, for almost every $\omega \in \Omega$,

$$\lim_{n \to \infty} \frac{s_n(\omega)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \theta^k(\omega) = f^*(\omega),$$

and, moreover, f^* is the conditional expectation of f given the σ-algebra of invariant sets over θ, i.e., for every set $A \in \mathfrak{F}$ with $\theta^{-1}(A) = A$,

$$\int_A f(\omega) d\mathbb{P}(\omega) = \int_A f^*(\omega) d\mathbb{P}(\omega).$$

Write $p = \alpha_1 p^1 + \dots + \alpha_R p^R$ as in Proposition 2.21. Since, for every $i \in \underline{R}$, the set Ω^i is invariant under θ, we have

$$\int_{\Omega^i} f(\omega) d\mathbb{P}(\omega) = \int_{\Omega^i} f^*(\omega) d\mathbb{P}(\omega).$$

By Corollary 2.23, θ is ergodic with respect to $\mathbb{P}^{p^i}$ for every $i \in \underline{R}$, and thus f^* is constant $\mathbb{P}^{p^i}$-almost everywhere on Ω; since $\mathbb{P}^{p^i}(\Omega^i) = 1$ and the restriction of $\mathbb{P}$ to Ω^i is precisely $\alpha_i \mathbb{P}^{p^i}$, one obtains that f^* is constant $\mathbb{P}$-almost everywhere in Ω^i. Hence, for every $i \in \underline{R}$ and almost every $\widetilde{\omega} \in \Omega^i$,

$$\alpha_i f^*(\widetilde{\omega}) = \int_{\Omega^i} f(\omega) d\mathbb{P}(\omega) = \alpha_i \int_{\Omega^i} f(\omega) d\mathbb{P}^{p^i}(\omega) = \alpha_i \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t),$$

which proves (2.28). Since its right-hand side is a positive real number and the sets Ω^i for which $\mathbb{P}(\Omega^i) \neq 0$ cover Ω except for a set of measure zero, the regularity of $(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))$ for almost every $\omega \in \Omega$ follows. ■

An immediate consequence of Theorem 2.11 and Proposition 2.25 is the following.

Theorem 2.26. *For every $x_0 \in \mathbb{R}^d \setminus \{0\}$ and almost every $\omega \in \Omega$, the Lyapunov exponents of the continuous- and discrete-time systems (2.7) and (2.17), given by (2.18), are related by*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega)) \lambda_{\mathrm{rc}}(x_0, \omega).$$

As a final result in this section, we prove the following proposition, which evaluates the average time spent in a certain state k.

Proposition 2.27. *Let $k \in \underline{N}$. For every $i \in \underline{R}$ such that $\mathbb{P}(\Omega^i) \neq 0$ and almost every $\omega \in \Omega^i$,*

$$\lim_{T \to \infty} \frac{\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = k\}}{T} = \frac{\chi_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t d\mu_k(t)}{\sum_{j \in C_i}^{N} p_j^i \int_{\mathbb{R}_+} t d\mu_j(t)},$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}$.

Proof. Fix $k \in \underline{N}$. Let $\varphi_k : \Omega \to \mathbb{R}_+$ be given by

$$\varphi_k((i_n, t_n)_{n=1}^\infty) = \begin{cases} t_1, & \text{if } i_1 = k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Birkhoff's Ergodic Theorem, there exists a function $\varphi_k^* \in L^1(\Omega, \mathbb{R}_+)$ invariant under θ such that, for almost every $\omega \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) = \varphi_k^*(\omega), \tag{2.29}$$

and, for every $i \in \underline{R}$,

$$\int_{\Omega^i} \varphi_k(\widetilde{\omega}) d\mathbb{P}(\widetilde{\omega}) = \int_{\Omega^i} \varphi_k^*(\widetilde{\omega}) d\mathbb{P}(\widetilde{\omega}).$$

As in the proof of Proposition 2.25, one shows that, for every $i \in \underline{R}$, φ_k^* is constant almost everywhere on Ω^i. Writing $p = \alpha_1 p^1 + \cdots + \alpha_R p^R$ as in Proposition 2.21, we get, for $i \in \underline{R}$ and almost every $\omega \in \Omega^i$,

$$\alpha_i \varphi_k^*(\omega) = \int_{\Omega^i} \varphi_k(\widetilde{\omega}) d\mathbb{P}(\widetilde{\omega}) = \alpha_i \sum_{j \in C_i} p_j^i \int_{\mathbb{R}_+} t \delta_{jk} d\mu_j(t) = \alpha_i \chi_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t d\mu_k(t). \tag{2.30}$$

By definition of $\boldsymbol{\alpha}$, for every $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$,

$$\sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) = \sum_{\substack{j=1\\ i_j=k}}^{n} t_j = \mathcal{L}\{t \in [0, s_n(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = k\}.$$

Hence, using Proposition 2.25 and combining (2.29) and (2.30), we obtain that, for every $i \in \underline{R}$ with $\mathbb{P}(\Omega^i) \neq 0$ and almost every $\omega \in \Omega^i$,

$$\lim_{n\to\infty} \frac{\mathcal{L}\{t \in [0, s_n(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = k\}}{s_n(\omega)} = \lim_{n\to\infty} \frac{n}{s_n(\omega)} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k(\theta^j \omega) = \frac{\chi_{C_i}(k) p_k^i \int_{\mathbb{R}_+} t\, d\mu_k(t)}{\sum_{j\in C_i} p_j^i \int_{\mathbb{R}_+} t\, d\mu_j(t)}. \tag{2.31}$$

Let $\omega \in \Omega$ be such that (2.31) holds and take $T \in \mathbb{R}_+$. Choose $n_T \in \mathbb{N}$ such that $s_{n_T}(\omega) \leq T < s_{n_T+1}(\omega)$. Then

$$\frac{1}{T}\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = k\} \leq \frac{1}{s_{n_T}(\omega)}\mathcal{L}\{t \in [0, s_{n_T+1}(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = k\}$$

and

$$\frac{1}{T}\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = k\} \geq \frac{1}{s_{n_T+1}(\omega)}\mathcal{L}\{t \in [0, s_{n_T}(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = k\}.$$

The conclusion of the proposition then follows since, by Proposition 2.25, $\frac{s_{n+1}(\omega)}{s_n(\omega)} \to 1$ as $n \to \infty$ for almost every $\omega \in \Omega$. ■

Remark 2.28. The choice of the compatible sequence in this section is not unique, and one might be interested in other possible choices. The times $s_n(\omega)$ in the sequence $\mathbf{s}(\omega)$ correspond to the transitions of the Markov chain from Proposition 2.2. However, if some of the diagonal elements of M are non-zero, then the discrete part of the Markov chain, i.e., its component in $\underline{N}$, may switch from a certain state to itself. In practical situations, it may be possible to observe only switches between different states, and another possible choice for the sequence $\mathbf{s}(\omega)$ that may be of practical interest is to consider only the times corresponding to such non-trivial switches. This can be done if $M_{ii} \neq 1$ for every $i \in \underline{N}$, i.e., if the Markov chain in the discrete space $\underline{N}$ has no absorbing states, in which case we have almost surely an infinite number of switches between different states. Defining θ as the shift to the next different state, θ defines a metric dynamical system if we suppose that, instead of having $pM = p$, we have $p\tilde{M} = p$, where $\tilde{M}_{ij} = \frac{M_{ij}}{1-M_{ii}}$ for $i, j \in \underline{N}$ with $i \neq j$ and $\tilde{M}_{ii} = 0$ for $i \in \underline{N}$. The counterparts of the previous results can be proved in this framework with no extra difficulty.

Remark 2.29. Even though we only consider in this chapter the case where p is invariant under M, our results can be generalized to the case of any probability vector p by using the following three facts. First, for any probability vector $q \in [0,1]^N$, the Cesàro mean of the sequence $(qM^n)_{n=0}^{\infty}$, namely $\frac{1}{n}\sum_{j=0}^{n-1} qM^j$, converges as $n \to \infty$ to an invariant probability vector (see, e.g., [131, Chapter 8]). Secondly, if $(q_n)_{n=1}^{\infty}$ is a sequence of probability vectors in $[0,1]^N$ converging to some probability vector $q \in [0,1]^N$, then $\mathbb{P}^{q_n}(E) \to \mathbb{P}^q(E)$ uniformly in $E \in \mathfrak{F}$ (which can be shown directly from (2.20)). Finally, if $q \in [0,1]^N$ is a probability vector and $q_0 = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} qM^j$, then $\mathbb{P}^q(E) = \mathbb{P}^{q_0}(E)$ for every set $E \in \mathfrak{F}$ invariant under θ (which follows from the fact that $\mathbb{P}^q(\theta^{-1}(E)) = \mathbb{P}^{qM}(E)$ for every $E \in \mathfrak{F}$). With these three properties, when the probability vector $p \in [0,1]^N$ is not invariant under M, it can be replaced in the previous results by the invariant probability vector given by the Cesàro mean $q = \frac{1}{n}\sum_{j=0}^{n-1} pM^j$ and the proofs can be adapted accordingly without much extra effort.

Remark 2.30. The fact that systems (2.1) and (2.8) are linear has been used only in the proof of Theorem 2.11, where one uses an exponential bound on the growth of the flows $\Phi_t^i = e^{A_i t}$, namely that there exist constants $C, \gamma > 0$ such that $\left|e^{A_i t}\right| \le Ce^{\gamma t}$ for every $t \ge 0$ and $i \in \underline{N}$. If we consider, instead of system (2.1), the nonlinear switched system

$$\dot{x}(t) = f_{\alpha(t)}(x(t)),$$

where $f_1, \dots, f_N$ are complete vector fields generating flows $\Phi^1, \dots, \Phi^N$, and modify the discrete-time system (2.8) accordingly, all the previous results remain true, with the same proofs, under the additional assumption that there exist constants $C, \gamma > 0$ such that $\left|\Phi_t^i(x)\right| \le Ce^{\gamma t}|x|$ for every $t \ge 0$, $i \in \underline{N}$, and $x \in \mathbb{R}^d$. However, the results from the next sections do not generalize to the nonlinear framework.

2.4 Multiplicative Ergodic Theorem

In this section, we apply the discrete-time Oseledets' Multiplicative Ergodic Theorem (see, e.g., [13, Theorem 3.4.1]) in the one-sided invertible case to system (2.17) and we use Proposition 2.25 and Theorem 2.26 to obtain that several of its conclusions also hold for the continuous-time system (2.7).

Recall that, for $i \in \underline{N}$, we consider $A_i \in \mathcal{M}_d(\mathbb{R})$ and $\Phi_t^i = e^{A_i t}$. Let $A : \Omega \to \mathcal{M}_d(\mathbb{R})$ be the function defined for $\omega = (i_n, t_n)_{n=1}^\infty$ by $A(\omega) = e^{A_{i_1} t_1}$, so that $\varphi_{\mathrm{rd}}(n; x_0, \omega) = A(\theta^{n-1}\omega)\varphi_{\mathrm{rd}}(n-1; x_0, \omega)$ for every $x_0 \in \mathbb{R}^d$, $\omega \in \Omega_0$, and $n \in \mathbb{N}^*$. For $\omega \in \Omega_0$ and $n \in \mathbb{N}$, we denote $\mathit{\Phi}(n, \omega)$ the linear operator defined by $\mathit{\Phi}(n, \omega)x = \varphi_{\mathrm{rd}}(n; x, \omega)$ for every $x \in \mathbb{R}^d$, which is thus given by $\mathit{\Phi}(n, \omega) = e^{A_{i_n} t_n} \cdots e^{A_{i_1} t_1}$ for $\omega = (i_j, t_j)_{j=1}^\infty \in \Omega_0$ and $n \in \mathbb{N}^*$.

Theorem 2.31. *There exists an invariant measurable subset $\widehat{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure such that, for every $\omega \in \widehat{\Omega}$,*

(a) *the limit $\Psi(\omega) = \lim_{n\to\infty} \left(\mathit{\Phi}(n, \omega)^{\mathrm{T}} \mathit{\Phi}(n, \omega)\right)^{1/2n}$ exists and is a positive definite matrix;*

(b) *there exist an integer $q(\omega) \in \underline{d}$ and $q(\omega)$ vector subspaces $V_1(\omega), \dots, V_{q(\omega)}(\omega)$ with respective dimensions $d_1(\omega) > \cdots > d_{q(\omega)}(\omega)$ such that*

$$V_{q(\omega)}(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d,$$

and $A(\omega)V_i(\omega) = V_i(\theta(\omega))$ for every $i \in \underline{q(\omega)}$;

(c) *for every $x_0 \in \mathbb{R}^d \setminus \{0\}$, the Lyapunov exponents $\lambda_{\mathrm{rd}}(x_0, \omega)$ and $\lambda_{\mathrm{rc}}(x_0, \omega)$ exist as limits, i.e.,*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \lim_{n\to\infty} \frac{1}{n} \log|\varphi_{\mathrm{rd}}(n; x_0, \omega)|,$$
$$\lambda_{\mathrm{rc}}(x_0, \omega) = \lim_{t\to\infty} \frac{1}{t} \log|\varphi_{\mathrm{rc}}(t; x_0, \omega)|;$$

(d) *there exist real numbers $\lambda_1^{\mathrm{d}}(\omega) > \cdots > \lambda_{q(\omega)}^{\mathrm{d}}(\omega)$ and $\lambda_1^{\mathrm{c}}(\omega) > \cdots > \lambda_{q(\omega)}^{\mathrm{c}}(\omega)$ such that, for every $i \in \underline{q(\omega)}$,*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \lambda_i^{\mathrm{d}}(\omega) \iff \lambda_{\mathrm{rc}}(x_0, \omega) = \lambda_i^{\mathrm{c}}(\omega) \iff x_0 \in V_i(\omega) \setminus V_{i+1}(\omega),$$

where $V_{q(\omega)+1}(\omega) = \{0\}$;

(e) *the eigenvalues of* $\Psi(\omega)$ *are* $e^{\lambda_1^{\mathrm{d}}(\omega)} > \cdots > e^{\lambda_{q(\omega)}^{\mathrm{d}}(\omega)}$;

(f) $q(\theta(\omega)) = q(\omega)$ *and, for* $i \in \underline{q(\omega)}$, $d_i(\theta(\omega)) = d_i(\omega)$, $\lambda_i^{\mathrm{d}}(\theta(\omega)) = \lambda_i^{\mathrm{d}}(\omega)$, *and* $\lambda_i^{\mathrm{c}}(\theta(\omega)) = \lambda_i^{\mathrm{c}}(\omega)$;

(g) *if* θ *is ergodic,* q *is constant on* $\widehat{\Omega}$, *and so are* d_i, λ_i^{d}, *and* λ_i^{c} *for* $i \in \underline{q}$.

Proof. Let us show that Oseledets' Multiplicative Ergodic Theorem can be applied to the random dynamical system $(\theta, \varphi_{\mathrm{rd}})$. Recall that there are $C \geq 1$, $\gamma > 0$ such that, for every $i \in \underline{N}$ and $t \in \mathbb{R}$, $\left|e^{A_i t}\right| \leq C e^{\gamma|t|}$. Then, for $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$, $\log^+ \left|A(\omega)^{\pm 1}\right| \leq \log C + \gamma t_1$, so that

$$\int_{\Omega} \log^+ \left|A(\omega)^{\pm 1}\right| d\mathbb{P}(\omega) \leq \log C + \gamma \sum_{i=1}^{N} p_i \int_{\mathbb{R}_+} t\, d\mu_i(t) < \infty.$$

Then Oseledets' Multiplicative Ergodic Theorem can be applied to (θ, φ_{rd}), yielding all the conclusions for Ψ, q, d_i, V_i, $\lambda_{\mathrm{rd}}(x_0, \omega)$, and λ_i^{d}. The conclusions concerning $\lambda_{\mathrm{rc}}(x_0, \omega)$ and $\lambda_i^{\mathrm{c}}(\omega)$ in (d), (f), and (g) follow from Theorem 2.26, with $\lambda_i^{\mathrm{c}}(\omega) = \frac{\lambda_i^{\mathrm{d}}(\omega)}{m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))}$. One is now left to show that the Lyapunov exponent $\lambda_{\mathrm{rc}}(x_0, \omega)$ exists as a limit.

Notice that $\left|e^{-A_i t} x\right| \leq C e^{\gamma t} |x|$ for every $i \in \underline{N}$, $x \in \mathbb{R}^d$ and $t \geq 0$, and hence $\left|e^{A_i t} x\right| \geq C^{-1} e^{-\gamma t} |x|$. Let $t > 0$ and choose $n_t \in \mathbb{N}$ such that $t \in \left(s_{n_t}(\omega), s_{n_t+1}(\omega)\right]$. Then, proceeding as in (2.15), one gets

$$\frac{1}{t} \log |\varphi_{\mathrm{rc}}(t; x_0, \omega)| \geq -\frac{\log C}{t} - \gamma \frac{t - s_{n_t}}{t} + \frac{1}{t} \log |\varphi_{\mathrm{rd}}(n_t; x_0, \omega)|.$$

Using (2.16), we thus obtain that

$$\liminf_{t \to \infty} \frac{1}{t} \log |\varphi_{\mathrm{rc}}(t; x_0, \omega)| \geq \frac{1}{m(\boldsymbol{\alpha}(\omega), \mathbf{s}(\omega))} \lambda_{\mathrm{rd}}(x_0, \omega) = \lambda_{\mathrm{rc}}(x_0, \omega),$$

which yields the existence of the limit. ■

2.5 The maximal Lyapunov exponent

We are interested in this section in the maximal Lyapunov exponents for systems (2.7) and (2.17), i.e., the real numbers $\lambda_1^{\mathrm{c}}(\omega)$ and $\lambda_1^{\mathrm{d}}(\omega)$ from Theorem 2.31(d). We denote these numbers by $\lambda_{\max}^{\mathrm{c}}(\omega)$ and $\lambda_{\max}^{\mathrm{d}}(\omega)$, respectively. Before proving the main results of this section, we state the following lemma, which shows that the Gelfand formula for the spectral radius ρ holds uniformly over compact sets of matrices. This follows from the estimates derived in [73, Section 3.3]. For the reader's convenience, we provide a proof.

Lemma 2.32. *Let* $\mathcal{A} \subset \mathcal{M}_d(\mathbb{R})$ *be a compact set of matrices. Then the limit*

$$\lim_{n \to \infty} |A^n|^{1/n} = \rho(A)$$

is uniform over $\mathcal{A}$.

Proof. Let $\varepsilon > 0$ and define a continuous function $F : \mathcal{A} \to \mathcal{M}_d(\mathbb{R})$ by

$$F(A) = \frac{1}{\rho(A) + \varepsilon} A.$$

Then $F(\mathcal{A})$ is compact and for every $F(A) \in F(\mathcal{A})$ its spectral radius is $\rho(F(A)) = \frac{\rho(A)}{\rho(A)+\varepsilon} < 1$. Fix $A \in \mathcal{A}$. Then there is a norm $|\cdot|_A$ in $\mathbb{R}^d$ with $|F(A)|_A < \frac{1+\rho(F(A))}{2}$ (see, e.g., [97, Lemma 5.6.10]). Hence, for all B in a neighborhood U of A

$$|F(B)|_A < \frac{1+\rho(F(A))}{2}.$$

Since all norms on $\mathcal{M}_d(\mathbb{R})$ are equivalent, there is $\beta_A > 0$ such that for all $B \in U$

$$|F(B)^n| \leq \beta_A |F(B)^n|_A \leq \beta_A |F(B)|_A^n \leq \beta_A \left(\frac{1+\rho(F(A))}{2}\right)^n.$$

Then there is $N \in \mathbb{N}^*$, depending only on A and ε, such that for all $n \geq N$ and all $B \in U$,

$$\frac{1}{\rho(B)+\varepsilon} |B^n|^{1/n} = |F(B)^n|^{1/n} < 1,$$

implying $|B^n|^{1/n} < \rho(B)+\varepsilon$. Since this holds for every B in a neighborhood U of A and $|B^n|^{1/n} \geq \rho(B)$ for every $n \in \mathbb{N}^*$, one obtains the uniformity of the convergence in U, and the assertion follows by compactness of $\mathcal{A}$. ∎

We can now prove our first result regarding the characterization of $\lambda_{\max}^{\mathrm{c}}$ and $\lambda_{\max}^{\mathrm{d}}$.

Theorem 2.33. *For almost every $\omega \in \Omega$, we have*

$$\lambda_{\max}^{\mathrm{d}}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log |\Phi(n,\omega)|. \tag{2.32}$$

If θ is ergodic, then $\lambda_{\max}^{\mathrm{d}}$ is constant almost everywhere and its constant value satisfies

$$\lambda_{\max}^{\mathrm{d}} \leq \inf_{n\in\mathbb{N}^*} \frac{1}{n} \int_\Omega \log|\Phi(n,\omega)|\, d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log|\Phi(n,\omega)|\, d\mathbb{P}(\omega). \tag{2.33}$$

Proof. Notice that (2.32) and (2.33) do not depend on the norm in $\mathcal{M}_d(\mathbb{R})$. We fix in this proof the norm induced by the Euclidean norm in $\mathbb{R}^d$, given by $|A| = \sqrt{\rho(A^{\mathrm{T}}A)}$. Notice that, in this case, $\left|A^{\mathrm{T}}A\right| = \sqrt{\rho((A^{\mathrm{T}}A)^2)} = \rho(A^{\mathrm{T}}A) = |A|^2$.

By Theorem 2.31(e), $e^{\lambda_{\max}^{\mathrm{d}}(\omega)}$ is the spectral radius $\rho(\Psi(\omega))$ of $\Psi(\omega)$. Using the continuity of the spectral radius and Theorem 2.31(a), one then gets that

$$e^{\lambda_{\max}^{\mathrm{d}}(\omega)} = \lim_{n\to\infty} \rho\left[\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{1/2n}\right].$$

By Gelfand's Formula for the spectral radius,

$$e^{\lambda_{\max}^{\mathrm{d}}(\omega)} = \lim_{n\to\infty}\lim_{k\to\infty} \left|\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{k/2n}\right|^{1/k}. \tag{2.34}$$

The sequence of matrices $\left(\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{1/2n}\right)_{n=1}^{\infty}$ converges to $\Psi(\omega)$, hence this sequence is bounded in $\mathcal{M}_d(\mathbb{R})$. By Lemma 2.32, the limit in Gelfand's Formula is uniform, which shows that one can take the limit in (2.34) along the line $k = 2n$ to obtain

$$e^{\lambda_{\max}^{\mathrm{d}}(\omega)} = \lim_{n\to\infty} \left|\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right|^{1/2n} = \lim_{n\to\infty} |\Phi(n,\omega)|^{1/n}.$$

Hence (2.32) follows by taking the logarithm.

If θ is ergodic, then, by Theorem 2.31(g), $\lambda_{\max}^{\mathrm{d}}$ is constant almost everywhere. Let $m \in \mathbb{N}^*$. By (2.32), for almost every $\omega \in \Omega$,

$$\lambda_{\max}^{\mathrm{d}} = \lim_{n \to \infty} \frac{1}{nm} \log \lvert \Phi(nm, \omega) \rvert. \tag{2.35}$$

One has $\Phi(nm, \omega) = \Phi(m, \theta^{(n-1)m}\omega) \cdots \Phi(m, \theta^m \omega)\Phi(m, \omega)$, and thus

$$\frac{1}{nm} \log \lvert \Phi(nm, \omega) \rvert \leq \frac{1}{nm} \sum_{k=0}^{n-1} \log \left\lvert \Phi(m, \theta^{mk}\omega) \right\rvert. \tag{2.36}$$

Since θ^m preserves $\mathbb{P}$ and $\log\lvert\Phi(m, \cdot)\rvert \in L^1(\Omega, \mathbb{R})$, Birkhoff's Ergodic Theorem shows that

$$\lim_{n \to \infty} \frac{1}{nm} \sum_{k=0}^{n-1} \log \left\lvert \Phi(m, \theta^{mk}\omega) \right\rvert = \frac{1}{m} \int_\Omega \log \lvert \Phi(m, \omega) \rvert \, d\mathbb{P}(\omega). \tag{2.37}$$

Combining (2.35), (2.36), and (2.37), one obtains the inequality in (2.33). The sequence $\left(\int_\Omega \log\lvert\Phi(n,\omega)\rvert \, d\mathbb{P}(\omega)\right)_n$ is subadditive, since $\Phi(n+m, \omega) = \Phi(m, \theta^n\omega)\Phi(n, \omega)$ for $n, m \in \mathbb{N}$ and θ preserves $\mathbb{P}$. This subadditivity implies that the equality in (2.33) holds. ∎

Under some extra assumptions on the probability measures μ_i, $i \in \underline{N}$, one obtains that the inequality in (2.33) is actually an equality.

Theorem 2.34. *Suppose that θ is ergodic and that there exists $r > 1$ such that, for every $i \in \underline{N}$, $\int_{(0,\infty)} t^r \, d\mu_i(t) < \infty$. Then $\lambda_{\max}^{\mathrm{d}}$ is constant almost everywhere and given by*

$$\lambda_{\max}^{\mathrm{d}} = \inf_{n \in \mathbb{N}^*} \frac{1}{n} \int_\Omega \log\lvert\Phi(n,\omega)\rvert \, d\mathbb{P}(\omega) = \lim_{n \to \infty} \frac{1}{n} \int_\Omega \log\lvert\Phi(n,\omega)\rvert \, d\mathbb{P}(\omega).$$

Proof. One clearly has, using (2.32), that

$$\lambda_{\max}^{\mathrm{d}} = \int_\Omega \lambda_{\max}^{\mathrm{d}}(\omega) d\mathbb{P}(\omega) = \int_\Omega \lim_{n \to \infty} \frac{1}{n} \log\lvert\Phi(n,\omega)\rvert \, d\mathbb{P}(\omega).$$

The theorem is proved if we show one can exchange the limit and the integral in the above expression, which we do by applying Vitali's convergence theorem (see, e.g., [152, Chapter 6]). We are thus left to show that the sequence of functions $\left(\frac{1}{n}\log\lvert\Phi(n,\cdot)\rvert\right)_{n=1}^\infty$ is uniformly integrable, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $E \in \mathfrak{F}$ with $\mathbb{P}(E) < \delta$, one has $\frac{1}{n}\left\lvert\int_E \log\lvert\Phi(n,\omega)\rvert \, d\mathbb{P}(\omega)\right\rvert < \varepsilon$.

For $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$ and $n \in \mathbb{N}^*$, one has $\Phi(n, \omega) = e^{A_{i_n} t_n} \cdots e^{A_{i_1} t_1}$. Let $C, \gamma > 0$ be such that $\left\lvert e^{A_i t}\right\rvert \leq C e^{\gamma t}$ for every $i \in \underline{N}$ and $t \geq 0$. Then

$$\log\lvert\Phi(n,\omega)\rvert \leq n \log C + \gamma \sum_{j=1}^n t_j = n \log C + \gamma s_n(\omega),$$

where $\mathbf{s}(\omega) = (s_n(\omega))_{n=0}^\infty$. Hence, it suffices to show that the sequence $\left(\frac{s_n}{n}\right)_{n=1}^\infty$ is uniformly integrable.

For $n \in \mathbb{N}^*$ and $E \in \mathfrak{F}$, we have, by Hölder's inequality,

$$\int_E \frac{s_n(\omega)}{n} d\mathbb{P}(\omega) = \frac{1}{n} \sum_{j=1}^n \int_E t_j \, d\mathbb{P}(\omega) \leq \frac{1}{n} \sum_{j=1}^n \left(\int_\Omega t_j^r \, d\mathbb{P}(\omega)\right)^{\frac{1}{r}} \mathbb{P}(E)^{\frac{1}{r'}} \leq K^{\frac{1}{r}} \mathbb{P}(E)^{\frac{1}{r'}}, \tag{2.38}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $K = \max_{i \in \underline{N}} \int_{(0,\infty)} t^r \, d\mu_i(t) < \infty$. Equation (2.38) establishes the uniform integrability of $\left(\frac{s_n}{n}\right)_{n=1}^\infty$, which yields the result. ∎

As an immediate consequence of Proposition 2.25, Theorem 2.26, Theorem 2.33, and Theorem 2.34, we obtain the following result.

Corollary 2.35. *Suppose that θ is ergodic. Then $\lambda_{\max}^{\mathrm{c}}$ and $\lambda_{\max}^{\mathrm{d}}$ are constants almost everywhere satisfying*

$$\lambda_{\max}^{\mathrm{d}} \leq \inf_{n\in\mathbb{N}^*} \frac{1}{n}\int_\Omega \log\lvert\Phi(n,\omega)\rvert\, d\mathbb{P}(\omega), \tag{2.39}$$

$$\lambda_{\max}^{\mathrm{c}} = \frac{\lambda_{\max}^{\mathrm{d}}}{\sum_{i=1}^N p_i \int_{\mathbb{R}_+} t\, d\mu_i(t)}.$$

In particular, if

$$\text{there exists } n\in\mathbb{N}^* \text{ such that } \int_\Omega \log\lvert\Phi(n,\omega)\rvert\, d\mathbb{P}(\omega) < 0, \tag{2.40}$$

then systems (2.7) and (2.17) are almost surely exponentially stable.

If we have further that there exists $r > 1$ such that $\int_{\mathbb{R}_+} t^r d\mu_i(t) < \infty$ for every $i \in \underline{N}$, then (2.39) is an equality and (2.40) is equivalent to the almost sure exponential stability of (2.7) and to the almost sure exponential stability of (2.17).

2.6 Application to the stabilization of control systems with arbitrary decay rate

In this section, we consider the linear control system

$$\dot{x}(t) = Ax(t) + B_{\alpha(t)}u_{\alpha(t)}(t), \tag{2.41}$$

where $x(t)\in\mathbb{R}^d$, $A\in\mathcal{M}_d(\mathbb{R})$, $\alpha:\mathbb{R}_+\to\underline{N}$ belongs to the class $\mathcal{P}$ of right continuous, piecewise constant switching signals, and, for $j\in\underline{N}$, $u_j(t)\in\mathbb{R}^{m_j}$ for some positive integer m_j and $B_j\in\mathcal{M}_{d,m_j}(\mathbb{R})$. System (2.41) is a switched control system with dynamics given by the N equations $\dot{x} = Ax + B_j u_j$, $j\in\underline{N}$.

Our main motivation to consider (2.41) comes from the analysis of persistently excited systems, described in Definition 1.2. In this framework, one is interested in stabilizing the system by a linear feedback $u = Kx$ with K depending on A, B, T, μ but chosen uniformly with respect to the (T,μ)-persistently exciting signal α. It is also of interest to determine the decay rates that can be achieved by such feedback laws K. In particular, as recalled in Proposition 1.23, [49, Proposition 4.5] shows that there are (two dimensional) controllable systems for which the achievable decay rates are bounded below, even when we consider only PE signals α taking values in $\{0,1\}$ instead of $[0,1]$. Our main result, Theorem 2.36, implies that, in the probabilistic setting defined below, one can get arbitrarily large (almost sure) decay rates for the generalization (2.41) of (1.7), which is in contrast to the situation for persistently excited systems. An explanation for this fact is that the probability of having a signal α with very fast switching for an infinitely long time, such as the signals used in the proof of [49, Proposition 4.5], is zero, and hence such signals do not interfere with the behavior of the (random) maximal Lyapunov exponent.

Let $M\in\mathcal{M}_N(\mathbb{R})$ be an irreducible stochastic matrix, p be its unique invariant probability vector, $\mu_1,\dots,\mu_N$ be probability measures on $\mathbb{R}_+^*$ with its Borel σ-algebra, and consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ from Definition 2.1. We consider system (2.41) in a probabilistic setting by taking random signals $\boldsymbol{\alpha}(\omega)$ as in Definition 2.5, i.e., the random control system

$\dot{x}(t) = Ax(t) + B_{\boldsymbol{\alpha}(\omega)(t)}u_{\boldsymbol{\alpha}(\omega)(t)}(t)$. The problem treated in this section is the arbitrary rate stabilizability of this system by linear feedback laws $u_j = K_j x$, $j \in \underline{N}$. More precisely, consider the closed-loop random switched system

$$\dot{x}(t) = \left(A + B_{\boldsymbol{\alpha}(\omega)(t)}K_{\boldsymbol{\alpha}(\omega)(t)}\right)x(t). \tag{2.42}$$

We wish to know if, given $\lambda \in \mathbb{R}$, there exist matrices $K_j \in \mathcal{M}_{m_j,d}(\mathbb{R})$, $j \in \underline{N}$, such that the maximal Lyapunov exponent $\lambda_{\max}^{\mathrm{c}}$ of the continuous-time system (2.42), defined as in Section 2.5, satisfies $\lambda_{\max}^{\mathrm{c}}(\omega) \leq \lambda$ for almost every $\omega \in \Omega$. Notice that, since we assume that M is irreducible, the discrete-time metric dynamical system θ defined in (2.19) is ergodic (see Remark 2.18), and hence, by Corollary 2.35, $\lambda_{\max}^{\mathrm{c}}$ is constant almost everywhere in Ω.

For $j \in \underline{N}$, let

$$\mathsf{V}_j = \operatorname{Ran}\begin{pmatrix} B_j & AB_j & \cdots & A^{d-1}B_j \end{pmatrix}. \tag{2.43}$$

Notice that, by Cayley–Hamilton theorem, for every $n \in \mathbb{N}$, all columns of $A^n B_j$ belong to V_j. Some of the spaces V_j may have dimension zero.

Theorem 2.36. *Let $A \in \mathcal{M}_d(\mathbb{R})$, $B_j \in \mathcal{M}_{d,m_j}(\mathbb{R})$ for $j \in \underline{N}$ and some $m_j \in \mathbb{N}^*$, and suppose that the spaces $\mathsf{V}_1,\dots,\mathsf{V}_N$ defined in (2.43) satisfy $\mathsf{V}_1 \oplus \cdots \oplus \mathsf{V}_N = \mathbb{R}^d$. Then, for every $\lambda \in \mathbb{R}$, there exist matrices $K_j \in \mathcal{M}_{m_j,d}(\mathbb{R})$, $j \in \underline{N}$, such that the maximal Lyapunov exponent $\lambda_{\max}^{\mathrm{c}}$ of the closed-loop random switched system (2.42) satisfies $\lambda_{\max}^{\mathrm{c}}(\omega) \leq \lambda$ for almost every $\omega \in \Omega$.*

Proof. For $j \in \underline{N}$, let $n_j = \dim \mathsf{V}_j$. Up to a linear change of variables in $\mathbb{R}^d$, we can suppose that $\mathsf{V}_1 = \{e_1,\dots,e_{n_1}\}, \mathsf{V}_2 = \{e_{n_1+1},\dots,e_{n_1+n_2}\},\dots,\mathsf{V}_N = \{e_{n_1+\dots+n_{N-1}+1},\dots,e_{n_1+\dots+n_N}\}$. In this case, for $j \in \underline{N}$, the matrices A and B_j have the block structure

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_N \end{pmatrix}, \qquad B_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b_j \\ \vdots \\ 0 \end{pmatrix}, \tag{2.44}$$

with $A_j \in \mathcal{M}_{n_j}(\mathbb{R})$ and $b_j \in \mathcal{M}_{n_j,m_j}(\mathbb{R})$. Whenever $n_j \neq 0$, it follows immediately from the definition of V_j that the pair (A_j, b_j) is controllable. Denoting by $P_j = (e_{n_1+\dots+n_{j-1}+1},\dots,e_{n_1+\dots+n_j})^{\mathrm{T}} \in \mathcal{M}_{n_j,d}(\mathbb{R})$, we have that $b_j = P_j B_j$ and $A_j = P_j A P_j^{\mathrm{T}}$.

Let $C \geq 1$, $\beta > 0$ be such that, for every $j \in \underline{N}$ and every $t \geq 0$, $\left|e^{A_j t}\right| \leq Ce^{\beta t}$. Thanks to [42, Proposition 2.1], we may assume that C is chosen large enough such that the following property holds: there exists $L \in \mathbb{N}^*$ such that, for every $\gamma \geq 1$ and $j \in \underline{N}$, there exists a matrix $k_j \in \mathcal{M}_{m_j,n_j}(\mathbb{R})$ with

$$\left|e^{(A_j+b_jk_j)t}\right| \leq C\gamma^L e^{-\gamma t}, \qquad \forall t \in \mathbb{R}_+. \tag{2.45}$$

Let $K_j = k_j P_j \in \mathcal{M}_{m_j,d}(\mathbb{R})$. With this choice of feedback laws, we have

$$A + B_jK_j = \begin{pmatrix} A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_j + b_jk_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_N \end{pmatrix},$$

and thus, for every $t \in \mathbb{R}$,

$$e^{(A+B_jK_j)t} = \begin{pmatrix} e^{A_1 t} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & e^{A_2 t} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(A_j+b_jk_j)t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & e^{A_N t} \end{pmatrix}.$$

Since M is irreducible and p is invariant under M, we have $p_j > 0$ for every $j \in \underline{N}$. The irreducibility of M also provides the existence of $r \geq N$ and $(i_1^*, \dots, i_r^*) \in \underline{N}^r$ such that $\{i_1^*, \dots, i_r^*\} = \underline{N}$ and $M_{i_1^* i_2^*} \cdots M_{i_{r-1}^* i_r^*} > 0$. In order to apply Corollary 2.35, consider

$$\begin{aligned} \int_\Omega \log|\Phi(r,\omega)|\, d\mathbb{P}(\omega) = & \sum_{(i_1,\dots,i_r)\in\underline{N}^r} p_{i_1} M_{i_1 i_2} \cdots M_{i_{r-1} i_r} \\ & \cdot \int_{(0,\infty)^r} \log\left|e^{(A+B_{i_r}K_{i_r})t_r} \cdots e^{(A+B_{i_1}K_{i_1})t_1}\right| d\mu_{i_1}(t_1) \cdots d\mu_{i_r}(t_r). \end{aligned} \tag{2.46}$$

Since $\sum_{j=1}^N P_j^{\mathrm{T}} P_j = \mathrm{Id}_d$ and $P_j e^{(A+B_iK_i)t} P_k^{\mathrm{T}} = 0$ if $j \neq k$, we have, for every $(i_1, \dots, i_r) \in \underline{N}^r$ and $(t_1, \dots, t_r) \in \mathbb{R}_+^r$,

$$\begin{aligned} e^{(A+B_{i_r}K_{i_r})t_r} \cdots e^{(A+B_{i_1}K_{i_1})t_1} &= \left(\sum_{j_r=1}^N P_{j_r}^{\mathrm{T}} P_{j_r}\right) e^{(A+B_{i_r}K_{i_r})t_r} \cdots \left(\sum_{j_1=1}^N P_{j_1}^{\mathrm{T}} P_{j_1}\right) e^{(A+B_{i_1}K_{i_1})t_1} \left(\sum_{j_0=1}^N P_{j_0}^{\mathrm{T}} P_{j_0}\right) \\ &= \sum_{j=1}^N P_j^{\mathrm{T}} P_j e^{(A+B_{i_r}K_{i_r})t_r} \cdots P_j^{\mathrm{T}} P_j e^{(A+B_{i_1}K_{i_1})t_1} P_j^{\mathrm{T}} P_j. \\ &= \sum_{j=1}^N P_j^{\mathrm{T}} e^{(A_j+\delta_{j i_r} b_j k_j)t_r} \cdots e^{(A_j+\delta_{j i_1} b_j k_j)t_1} P_j. \end{aligned} \tag{2.47}$$

Since, for every $j \in \underline{N}$ and $t \geq 0$, we have $\left|e^{A_j t}\right| \leq C e^{\beta t}$ and $\left|e^{(A_j+b_jk_j)t}\right| \leq C \gamma^L e^{-\gamma t}$, we get, for every $(i_1, \dots, i_r) \in \underline{N}^r$ and $(t_1, \dots, t_r) \in \mathbb{R}_+^r$,

$$\left|e^{(A+B_{i_r}K_{i_r})t_r} \cdots e^{(A+B_{i_1}K_{i_1})t_1}\right| \leq N C^r \gamma^{rL} e^{\beta \sum_{j=1}^r t_j}. \tag{2.48}$$

When $(i_1, \dots, i_r) = (i_1^*, \dots, i_r^*)$, we can obtain a sharper bound than (2.48). For $j \in \underline{N}$, denote by $N(j)$ the nonempty set of all indices $k \in \underline{r}$ such that $i_k^* = j$, and denote by $n(j) \in \mathbb{N}^*$ the number of elements in $N(j)$. Then

$$\left|P_j^{\mathrm{T}} e^{(A_j+\delta_{j i_r^*} b_j k_j)t_r} \cdots e^{(A_j+\delta_{j i_1^*} b_j k_j)t_1} P_j\right| \leq C^r \gamma^{n(j)L} e^{-\gamma \sum_{k\in N(j)} t_k} e^{\beta \sum_{k \in \underline{r}\setminus N(j)} t_k},$$

which shows, using (2.47), that

$$\begin{aligned} \left|e^{(A+B_{i_r^*}K_{i_r^*})t_r} \cdots e^{(A+B_{i_1^*}K_{i_1^*})t_1}\right| &\leq \sum_{j=1}^N C^r \gamma^{n(j)L} e^{-\gamma \sum_{k\in N(j)} t_k} e^{\beta \sum_{k\in\underline{r}\setminus N(j)} t_k} \\ &\leq N C^r \gamma^{rL} e^{-\gamma \min_{k\in\underline{r}} t_k} e^{r\beta \max_{k\in\underline{r}} t_k}. \end{aligned} \tag{2.49}$$

Let

$$\begin{aligned} E_0 &= \max_{i \in \underline{N}} \int_{(0,\infty)} t \, d\mu_i(t), \\ E_{\min} &= \int_{(0,\infty)^r} \min_{k \in \underline{r}} t_k \, d\mu_{i_1^*}(t_1) \cdots d\mu_{i_r^*}(t_r) > 0, \\ E_{\max} &= \int_{(0,\infty)^r} \max_{k \in \underline{r}} t_k \, d\mu_{i_1^*}(t_1) \cdots d\mu_{i_r^*}(t_r) < \infty. \end{aligned}$$

Then, combining (2.48) and (2.49), we obtain from (2.46) that

$$\begin{aligned} \int_\Omega \log\lvert\Phi(r,\omega)\rvert \, d\mathbb{P}(\omega) &\leq N^r \left(\log(NC^r) + rL\log\gamma + r\beta E_0\right) \\ &\quad + p_{i_1^*} M_{i_1^* i_2^*} \cdots M_{i_{r-1}^* i_r^*} \left(\log(NC^r) + rL\log\gamma - \gamma E_{\min} + r\beta E_{\max}\right). \end{aligned} \tag{2.50}$$

The right-hand side of (2.50) tends to $-\infty$ as $\gamma \to \infty$, which can be achieved by (2.45). Hence it follows from Corollary 2.35 that the maximal Lyapunov exponent of (2.42) can be made arbitrarily small. ∎

Remark 2.37. By writing the matrices A and B_j, $j \in \underline{N}$, in the form (2.44), system (2.41) can be seen as N independent control systems such that, at each time, only one of them is controlled, while the others follow their uncontrolled dynamics.

Remark 2.38. In order to establish a more precise link between Theorem 2.36 and the case of deterministic persistently excited systems treated in [39, 45, 46, 49, 128] and recalled in Section 1.2.1, consider the case of (2.41) with $\alpha(t) \in \{0,1\}$, $B_0 = 0$, $B_1 = B$, and (A, B) controllable. Moreover, in order to simplify, we assume that, in the probabilistic model of $\boldsymbol{\alpha}$, trivial switches do not occur, which amounts to choosing

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with unique invariant probability vector $p = \left(\frac{1}{2}, \frac{1}{2}\right)$. In general, such signals $\boldsymbol{\alpha}(\omega)$ cannot be persistently exciting. In fact, suppose that μ_0 satisfies $\mu_0((0,T]) < 1$ for every $T > 0$. Then

$$\mathbb{P}\{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \boldsymbol{\alpha}(\omega) \in \mathcal{G}(T,\mu)\} = 0. \tag{2.51}$$

Indeed, since a (T,μ)-persistently exciting signal is also a (T',μ')-persistently exciting signal for every $T' \geq T$ and $0 < \mu' \leq \mu$, we have

$$\begin{aligned} \{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \boldsymbol{\alpha}(\omega) \in \mathcal{G}(T,\mu)\} &= \bigcup_{T>0} \bigcup_{\mu \in (0,T]} \{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \in \mathcal{G}(T,\mu)\} \\ &= \bigcup_{T \in \mathbb{N}^*} \bigcup_{\frac{1}{\mu} \in \mathbb{N}^*} \{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \in \mathcal{G}(T,\mu)\}. \end{aligned}$$

If $\alpha \in \mathcal{G}(T,\mu)$, the persistence of excitation condition implies that α cannot remain zero during time intervals longer than $T - \mu$, and thus

$$\{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \in \mathcal{G}(T,\mu)\} \subset \{\omega = (i_n, t_n)_{n=1}^\infty \in \Omega \mid \forall n \in \mathbb{N}^*,\ i_n = 0 \implies t_n \leq T - \mu\}. \tag{2.52}$$

Since i_n takes the value 0 infinitely many times for almost every $\omega \in \Omega$ and $\mu_0((0, T-\mu]) < 1$, the right-hand side of (2.52) has measure zero, and thus (2.51) holds.

However, one can link the random signals $\boldsymbol{\alpha}(\omega)$ with a weaker, asymptotic notion of persistence of excitation. A (deterministic) measurable signal $\alpha : \mathbb{R}_+ \to [0,1]$ is said to be *asymptotically persistently exciting* with constant $\rho > 0$ if

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \alpha(s) ds \geq \rho.$$

It follows easily from (1.6) that every (T,μ)-persistently exciting signal is also asymptotically persistently exciting with constant $\rho = \frac{\mu}{T}$. Proposition 2.27 implies that, for almost every $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \boldsymbol{\alpha}(\omega)(s) ds = \frac{\int_{\mathbb{R}_+} t d\mu_1(t)}{\int_{\mathbb{R}_+} t d\mu_0(t) + \int_{\mathbb{R}_+} t d\mu_1(t)},$$

and thus, in particular, almost every signal $\boldsymbol{\alpha}(\omega)$ is asymptotically persistently exciting with constant $\rho = \frac{\int_{\mathbb{R}_+} t d\mu_1(t)}{\int_{\mathbb{R}_+} t d\mu_0(t) + \int_{\mathbb{R}_+} t d\mu_1(t)} > 0$.

Chapter 3

Persistently damped transport on a network of circles

3.1 Introduction

Consider the following system of $N \geq 2$ coupled transport equations,

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(x) u_i(t,x) = 0, & t \geq 0,\ x \in [0,L_i],\ i \in [\![1,N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0,L_i],\ i \in [\![N_d+1,N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^{N} m_{ij} u_j(t,L_j), & t \geq 0,\ i \in [\![1,N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in [\![1,N]\!]. \end{cases} \tag{3.1}$$

For $i \in [\![1,N]\!]$, the corresponding transport equation is defined in the space domain $[0,L_i]$ with $L_i > 0$. The integer N_d denotes the number of equations with a damping term. For $i \in [\![1,N_d]\!]$, the activity of the damping of the i-th equation in space is determined by the function χ_i, which is assumed to be the characteristic function of an interval $[a_i,b_i] \subset [0,L_i]$ with $a_i < b_i$, whereas its activity in time is determined by the function α_i, which is assumed to be a signal in $L^\infty(\mathbb{R},[0,1])$. The coupling between the N transport equations is determined by the coefficients $m_{ij} \in \mathbb{R}$ for $1 \leq i,j \leq N$. The goal of this chapter consists in studying the stability properties of (3.1) when the signals α_i are persistently exciting, as described in Definition 1.2.

System (3.1) is a system of N transport equations defined on intervals $[0,L_i]$, $1 \leq i \leq N$, which may be identified with circles $C_1, C_2, \dots, C_N$ of respective lengths $L_1, L_2, \dots, L_N$. Moreover, there exists a point O such that any two distinct circles only intersect at O (see Figure 3.1). The transmission condition at O can be written as

$$\begin{pmatrix} u_1(t,0) \\ u_2(t,0) \\ \vdots \\ u_N(t,0) \end{pmatrix} = M \begin{pmatrix} u_1(t,L_1) \\ u_2(t,L_2) \\ \vdots \\ u_N(t,L_N) \end{pmatrix}, \tag{3.2}$$

where $M = (m_{ij})_{i,j\in[\![1,N]\!]}$ is called the *transmission matrix* of the system. The topology of the network considered in this chapter is star-shaped with respect to the point O. Note that any other network configuration falls into the present framework by a suitable choice of transition matrix M, namely, the fact that two circles C_i and C_j are not inward-outward adjacent translates to $m_{ij} = 0$. For $i \in [\![1, N_d]\!]$, the transport equation of u_i is damped on the support $[a_i, b_i]$ of χ_i, represented in Figure 3.1. The damping is subject to the signal α_i, which can be zero on certain time intervals. When all the α_i take their values in $\{0, 1\}$, (3.1) can be seen as a switched system, where the switching signal α_i controls the damping action on the interval $[a_i, b_i]$ of the circle C_i.

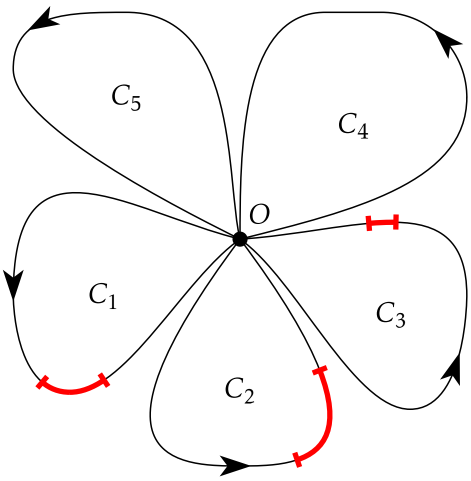


Figure 3.1: Network corresponding to $N = 5$ and $N_d = 3$.

Switching occurs in several control applications, motivating the study of systems with switched or intermittent actuators, as presented in Section 1.1. In this context, the activity of the actuator is guaranteed by appropriate conditions, for instance existence of a positive dwell-time or average dwell-time [113]. In this chapter we rely instead on the integral condition (1.6) to guarantee the damping activity. As recalled in Section 1.2.1, this condition finds its origin in problems of identification and adaptive control [9–11], where it is used in a more general form as a necessary and sufficient condition for the global exponential stability of some linear time-dependent systems. Persistently excited systems, as described in Definition 1.2, have been considered in the literature in the finite-dimensional setting in [38, 39, 45, 46, 49, 126, 128], dealing mostly with problems concerning stabilizability by a linear feedback law. In such systems, the persistently exciting signal α is a convenient tool to model several phenomena, such as failures in links between systems, resource allocation, or other internal or external processes that affect control efficiency. Their infinite-dimensional counterparts are much less present in the literature, due to the fact that finite-dimensional results cannot be straightforwardly generalized, as illustrated by Example 1.26.

System (3.1) is a "toy model" to study infinite-dimensional systems under persistent excitation. It is a simple case of a *multi-body structure*, as remarked in Section 1.3 (see also [1, 6, 24, 35, 110, 119] and references therein). Notice that (3.1) is related to systems of wave propagation on networks. Indeed, by decomposing each one-dimensional wave equation intro traveling waves according to D'Alembert decomposition, one can replace an edge of the graph by a pair of oriented edges and consider the transport equation in each edge. Hence, when (3.1) is undamped (i.e., when $N_d = 0$), it actually represents the D'Alembert decomposition of a star-shaped network of strings. The damping term in (3.1) does not come from the above decomposition of the wave equation and thus the results of this chapter cannot be directly applied to wave propagation on networks.

This chapter addresses the issue of exponential stability of (3.1), uniformly with respect to the signals α_i in a class $\mathcal{G}(T, \mu)$: given $T \geq \mu > 0$, is system (3.1) uniformly exponentially stable with respect to $\alpha_i \in \mathcal{G}(T, \mu)$, $i \in [\![1, N_d]\!]$? The answer clearly depends on the transmission matrix M, since this matrix can amplify or reduce the solutions when they pass through O, as well as on the rationality of the ratios L_i/L_j, since periodic solutions may exist when they are rational (see Sections 3.2.2 and 3.2.3 below). The main result of this chapter is the following.

Theorem 3.1. *Suppose that $N \geq 2$, $N_d \geq 1$, $|M|_{\ell^1} \leq 1$, $m_{ij} \neq 0$ for every $i, j \in [\![1, N]\!]$, and that there exist $i_*, j_* \in [\![1, N]\!]$ such that $L_{i_*}/L_{j_*} \notin \mathbb{Q}$. Then, for every $T \geq \mu > 0$, there exist $C, \gamma > 0$ such that, for every $p \in [1, +\infty]$, every initial condition $u_{i,0} \in L^p(0, L_i)$, $i \in [\![1, N]\!]$, and every choice of signals $\alpha_i \in \mathcal{G}(T, \mu)$, $i \in [\![1, N_d]\!]$, the corresponding solution of (3.1) satisfies*

$$\sum_{i=1}^{N} \|u_i(t)\|_{L^p(0,L_i)} \leq C e^{-\gamma t} \sum_{i=1}^{N} \left\|u_{i,0}\right\|_{L^p(0,L_i)}, \qquad \forall t \in \mathbb{R}_+.$$

Our argument is based on explicit formulas for the solutions of system (3.1), which allow one to efficiently track down the effects of the persistency of the damping. This approach can be worked out since system (3.1) consists of constant-speed transport equations with local damping. On the other hand, the usual techniques from PDE control, such as Carleman estimates, spectral criteria, Ingham estimates or microlocal analysis, do not seem well-adapted here, since they do not allow to handle the effects due to the time-dependency induced by the persistently exciting signals α_i. Extensions of our result to the case of state-dependent speed of transport and non-local damping would probably require more refined techniques.

The idea of relying on explicit representations for solutions of (3.1) to address control and identification issues has already been used in [77, 168]. Note, however, that, in these two references, rational dependence assumptions were necessary to derive tractable explicit formulas, which is not the case in this chapter.

The chapter is organized as follows. In Section 3.2, we give some definitions used through this chapter, discuss the well-posedness of (3.1), and explain the role of the hypotheses of Theorem 3.1. Section 3.3 provides the explicit representation formula for the solutions of (3.1), first in the undamped case, where the notations are simpler and the formulas easier to write, and then in the general case. Our main result is proved in Section 3.4, where we study the asymptotic behavior of coefficients appearing in the explicit solution obtained in Section 3.3. We finally collect in a series of appendices various technical results used in the chapter.

3.2 Definitions and preliminary facts

All Banach and Hilbert spaces considered in this chapter are supposed to be real. We shall refer to linear operators in a Banach space X simply as *operators*.

We refer to System (3.1) as being *undamped* by setting $\alpha_i \equiv 0$ for every $i \in [\![1, N_d]\!]$, in which case it is written as

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^{N} m_{ij} u_j(t, L_j), & t \in \mathbb{R}_+,\ i \in [\![1, N]\!], \\ u_i(0, x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!]. \end{cases} \tag{3.3}$$

We say that System (3.1) has an *always active damping* if $\alpha_i \equiv 1$ for every $i \in [\![1, N_d]\!]$, in which case it becomes

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \chi_i(x) u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t, L_j), & t \in \mathbb{R}_+,\ i \in [\![1, N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!]. \end{cases} \tag{3.4}$$

The general case of (3.1) can be written as

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t) \chi_i(t) u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in [\![1, N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij} u_j(t, L_j), & t \in \mathbb{R}_+,\ i \in [\![1, N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!], \end{cases} \tag{3.5}$$

with the convention that $\alpha_i \equiv 1$ and $a_i = b_i = L_i$ for $i \in [\![N_d + 1, N]\!]$, implying that $\chi_i = 0$ almost everywhere in $[0, L_i]$ for $i \in [\![N_d + 1, N]\!]$. In the case where $\alpha_1, \dots, \alpha_{N_d}$ belong to a class $\mathcal{G}(T, \mu)$ for the same fixed $T \geq \mu > 0$, (3.5) is referred to as a *persistently damped system*.

Remark 3.2. Assuming that all the persistently exciting signals $\alpha_1, \dots, \alpha_{N_d}$, in (3.5) belong to the class $\mathcal{G}(T, \mu)$, with the same constants $T \geq \mu > 0$, is not actually a restriction. Indeed, if $\alpha_i \in \mathcal{G}(T_i, \mu_i)$ with $T_i \geq \mu_i > 0$ for $i \in [\![1, N_d]\!]$, then we clearly have, for every $i \in [\![1, N_d]\!]$, $\alpha_i \in \mathcal{G}(T, \mu)$ with $T = \max_{i \in [\![1, N_d]\!]} T_i$ and $\mu = \min_{i \in [\![1, N_d]\!]} \mu_i$.

3.2.1 Formulation and well-posedness of the Cauchy problem

The goal of this section consists in providing a rigorous definition for a solution of (3.5) and in guaranteeing that, given any initial data, the required solution exists, is unique and depends continuously on the initial data.

Definition 3.3. Let $p \in [1, +\infty)$. We set $\mathsf{X}_p = \prod_{i=1}^N L^p(0, L_i)$, endowed with the usual norm $\|z\|_{\mathsf{X}_p} = \left(\sum_{i=1}^N \|u_i\|_{L^p(0,L_i)}^p\right)^{1/p}$ for $z = (u_1, \dots, u_N) \in \mathsf{X}_p$.

We define the operator $A : D(A) \subset \mathsf{X}_p \to \mathsf{X}_p$ on its domain $D(A)$ by

$$\begin{aligned} D(A) &= \left\{ (u_1, \dots, u_N) \in \prod_{i=1}^N W^{1,p}(0, L_i) \,\middle|\, \forall i \in [\![1, N]\!],\ u_i(0) = \sum_{j=1}^N m_{ij} u_j(L_j) \right\}, \\ A(u_1, \dots, u_N) &= \left(-\frac{du_1}{dx}, \dots, -\frac{du_N}{dx} \right). \end{aligned} \tag{3.6}$$

For $i \in [\![1, N_d]\!]$, we define the operator $B_i \in \mathcal{L}(\mathsf{X}_p)$ by

$$B_i(u_1, \dots, u_N) = (0, \dots, 0, -\chi_i u_i, 0, \dots, 0),$$

where the term $-\chi_i u_i$ is in the i-th position.

Remark 3.4. Even though Theorem 3.1 is stated for every $p \in [1, +\infty]$, we restrict ourselves in the sequel to the case $p \in [1, +\infty)$. The main reason for this is that, when $p = +\infty$, the domain $D(A)$ of the operator A defined by (3.6) is not dense in $\prod_{i=1}^N L^\infty(0, L_i)$, and thus some of our arguments given for p finite do not apply. However, once we prove Theorem 3.1 for $p \in [1, +\infty)$, we obtain the case $p = +\infty$ by suitable continuity arguments, as detailed in Remark 3.26.

With the operators A and B_i defined above, System (3.5) can be written as

$$\begin{cases} \dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t) B_i z(t), \\ z(0) = z_0, \end{cases} \tag{3.7}$$

with $z_0 = (u_{1,0}, \dots, u_{N,0})$ and $\alpha_1, \dots, \alpha_{N_d} \in L^\infty(\mathbb{R}, [0,1])$. The case of the undamped system (3.3) can be written as

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0, \end{cases} \tag{3.8}$$

and the system with an always active damping (3.4) becomes

$$\begin{cases} \dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} B_i z(t), \\ z(0) = z_0. \end{cases} \tag{3.9}$$

The well-posedness of (3.7) is established in the sense of the following theorem, whose proof is deferred in Appendix 3.A.

Theorem 3.5. *Let $p \in [1, +\infty)$ and $\alpha_i \in L^\infty(\mathbb{R}, [0,1])$ for $i \in [\![1, N_d]\!]$. There exists a unique evolution family $\{T(t,s)\}_{t \geq s \geq 0}$ of bounded operators in X_p such that, for every $s \geq 0$ and $z_0 \in D(A)$, $t \mapsto z(t) = T(t,s)z_0$ is the unique continuous function such that $z(s) = z_0$, $z(t) \in D(A)$ for every $t \geq s$, z is differentiable for almost every $t \geq s$, $\dot{z} \in L^\infty_{\mathrm{loc}}([s, +\infty), \mathsf{X}_p)$, and $\dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t) B_i z(t)$ for almost every $t \geq s$.*

The definition of an evolution family is recalled in Appendix 3.A. The function z in Theorem 3.5 is said to be a *regular solution* of (3.7) with initial condition $z_0 \in D(A)$. When $z_0 \in \mathsf{X}_p \setminus D(A)$, the function $t \mapsto z(t) = T(t,s)z_0$ is still well-defined and continuous, and is said to be a *mild solution* of (3.7). We use the word *solution* to refer to both regular and mild solutions, according to the context.

Theorem 3.5 also provides solutions to (3.8) and (3.9) as particular cases. Since these equations are time-independent, we can actually obtain more regular solutions, thanks to the fact that A and $A + \sum_{i=1}^{N_d} B_i$ generate strongly continuous semigroups, as we detail in Appendix 3.A.

Theorem 3.6. *Let $p \in [1, +\infty)$. The operators A and $A + \sum_{i=1}^{N_d} B_i$ generate strongly continuous semigroups $\{e^{tA}\}_{t \geq 0}$ and $\{e^{t(A + \sum_{i=1}^{N_d} B_i)}\}_{t \geq 0}$. In particular, for every $z_0 \in D(A)$, the function $t \mapsto e^{tA} z_0$ is the unique function in $\mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ satisfying (3.8) and the function $t \mapsto e^{t(A + \sum_{i=1}^{N_d} B_i)} z_0$ is the unique function in $\mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ satisfying (3.9).*

3.2.2 Some examples of asymptotic behavior

It is useful to have in mind some illustrative examples of the asymptotic behaviors of (3.1) under no damping, an always active damping and a persistent damping, respectively.

Example 3.7. Consider the case of a single transport equation on a circle of length L,

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) + \alpha(t)\chi(x) u(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L], \\ u(t,0) = u(t,L), & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in [0,L], \\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{3.10}$$

where χ is the characteristic function of the interval $[a,b] \subset [0,L]$. This corresponds to (3.5) with persistent damping, $N_d = N = 1$, and $m_{11} = 1$. Due to the condition $u(t,0) = u(t,L)$, it can be seen as a transport equation on a circle of length L.

When (3.10) is undamped, all its solutions are L-periodic. Indeed, for $u_0 \in \mathsf{X}_p = L^p(0,L)$, the corresponding solution of (3.10) is $u(t,x) = u_0(\{x-t\}_L)$, where we recall that $\{x\}_y = x - \lfloor x/y \rfloor y$, and this function is clearly L-periodic.

When (3.10) has an always active damping, all its solutions converge exponentially to zero. Indeed, every solution of (3.10) satisfies $u(t,x) = e^{-(b-a)} u(t-L,x)$ for every $x \in [0,L]$ and $t \geq L$, so that $\|u(t)\|_{L^p(0,L)} = e^{-(b-a)} \|u(t-L)\|_{L^p(0,L)}$. It is also clear that $\|u(t)\|_{L^p(0,L)} \leq \|u_0\|_{L^p(0,L)}$ for every $t \geq 0$, and so $\|u(t)\|_{L^p(0,L)} \leq C e^{-\gamma t} \|u_0\|_{L^p(0,L)}$ for every $t \geq 0$, with $\gamma = \frac{b-a}{L}$ and $C = e^{\gamma L}$.

When (3.10) has a persistent damping and the damping interval $[a,b]$ is a proper subset of $[0,L]$, there exist $T > \mu > 0$, a persistently exciting signal $\alpha \in \mathcal{G}(T,\mu)$ and a nontrivial initial condition $u_0 \in L^p(0,L)$ such that the corresponding solution of (3.10) is L-periodic. Indeed, suppose that $a = 0$ and $b < L$. Take $u_0 \in \mathcal{C}^\infty([0,L]) \setminus \{0\}$ such that the support of u_0 is contained in $[b, \frac{b+L}{2}]$. Take $\alpha \in L^\infty(\mathbb{R},[0,1])$ defined by

$$\alpha(t) = \begin{cases} 1, & \text{if } 0 \leq \{t\}_L \leq \frac{L-b}{2}, \\ 0, & \text{if } \{t\}_L > \frac{L-b}{2}. \end{cases}$$

Then $\alpha \in \mathcal{G}\left(L, \frac{L-b}{2}\right)$ and one can easily verify that the corresponding solution $u(t,x)$ of (3.10) is equal to $u_0(\{x-t\}_L)$. Hence (3.10) admits a L-periodic solution.

Example 3.7 shows that the asymptotic behavior of (3.5) can be different if the damping is always active or if it is submitted to a persistently exciting signal, and this is due to the fact that the support of the solution may not be in the damping interval $[a,b]$ when the damping is active. Notice that Example 3.7 can be seen as a version of Example 1.26 in the framework of transport equations.

We now consider a second example showing that, when we have more than one circle, the rationality of the ratios L_i/L_j for $i \neq j$ plays an important role in the asymptotic behavior.

Example 3.8. Consider the case of System (3.5) with persistent damping, $N = 2$, $N_d = 1$, and $m_{ij} = 1/2$ for $i,j \in \{1,2\}$, i.e.,

$$\begin{cases} \partial_t u_1(t,x) + \partial_x u_1(t,x) + \alpha(t)\chi(x) u_1(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_1], \\ \partial_t u_2(t,x) + \partial_x u_2(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0,L_2], \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in \{1,2\}, \\ \alpha \in \mathcal{G}(T,\mu), \end{cases} \tag{3.11}$$

where χ is the characteristic function of the interval $[a,b] \subset [0,L_1]$. In order to simplify the discussion, let us fix $p = 2$ and set $\mathsf{X}_2 = L^2(0,L_1) \times L^2(0,L_2)$.

When (3.11) is undamped its asymptotic behavior depends on the rationality of the ratio L_1/L_2, as stated in the next theorem, which is proved in Appendix 3.B.

Theorem 3.9. *Consider* (3.11) *with* $\chi \equiv 0$.

(a) *If* $L_1/L_2 \notin \mathbb{Q}$, *each solution converges to a constant function* $(\lambda,\lambda) \in \mathsf{X}_2$ *with* $\lambda \in \mathbb{R}$.

(b) *If* $L_1/L_2 \in \mathbb{Q}$, *there exists a non-constant periodic solution.*

When (3.11) has an always active damping all solutions converge exponentially to zero, independently of the rationality of the ratio L_1/L_2, as it follows, for instance, from Remark 3.33.

When (3.11) has a persistent damping, the rationality of the ratio L_1/L_2 plays once again a role in the asymptotic behavior of the system: if $L_1/L_2 \notin \mathbb{Q}$, all its solutions converge exponentially to zero, as it follows from our main result, Theorem 3.1. However, if $L_1/L_2 \in \mathbb{Q}$ and the damping interval $[a, b]$ is small enough, there exist $T > \mu > 0$, a persistently exciting signal $\alpha \in \mathcal{G}(T, \mu)$ and a nontrivial initial condition $u_0 \in L^p(0, L)$ such that the corresponding solution of (3.11) is periodic, as we show in Appendix 3.B.4.

Both in Example 3.7 and in Example 3.8 in the case $L_1/L_2 \in \mathbb{Q}$, the lack of exponential stability is illustrated by the existence of a periodic solution for the persistently damped system which is actually a solution to the undamped one for which a persistently exciting signal α inactivates the damping whenever the support of the solution passes through the damping interval. The heuristic of the proof of Theorem 3.1 is that, under an irrationality hypothesis, the support of every initial condition spreads with time and eventually covers the entire network. Hence every solution of the persistently excited system eventually passes through a damping interval at a time where the damping is active.

3.2.3 Discussion on the hypotheses of Theorem 3.1

Recall that the two main assumptions of Theorem 3.1 are the following.

Hypothesis 3.10. There exist $i_*, j_* \in [\![1, N]\!]$ such that $L_{i_*}/L_{j_*} \notin \mathbb{Q}$.

Hypothesis 3.11. The matrix M satisfies $|M|_{\ell^1} \leq 1$ and $m_{ij} \neq 0$ for every $i, j \in [\![1, N]\!]$.

At the light of Example 3.8, one cannot expect exponential stability of (3.5) with persistent damping in general if $L_i/L_j \in \mathbb{Q}$ for every $i, j \in [\![1, N]\!]$. This is why it is reasonable to make Hypothesis 3.10.

Even though the well-posedness of (3.5) discussed in Section 3.2.1 and the explicit formula for its solutions given later in Section 3.3 are obtained for every $M \in \mathcal{M}_N(\mathbb{R})$, the asymptotic behavior of System (3.5) clearly depends on the choice of the matrix M, since this matrix determines the coupling among the N transport equations.

The hypothesis $|M|_{\ell^1} \leq 1$ can be written as

$$\sum_{i=1}^{N} \left|m_{ij}\right| \leq 1, \qquad \forall j \in [\![1, N]\!]. \tag{3.12}$$

The coefficient m_{ij} can be interpreted as the proportion of mass in the circle C_j that goes to the circle C_i as it passes the contact point O. Hence, (3.12) states that, for every $j \in [\![1, N]\!]$, the total mass arriving at the circles C_i, $i \in [\![1, N]\!]$, from the circle C_j is less than or equal the total mass leaving the circle C_j, which means that the mass never increases while passing through the junction.

The hypothesis $m_{ij} \neq 0$ for all i, j can be seen as a *strong mixing* of the solutions at the junction. It is designed to avoid reducibility phenomena which may be an obstruction to uniform exponential stability. Consider for instance the case $M = \mathrm{Id}_N$ with $N \geq 2$. Then (3.5) is reduced to N uncoupled transport equations on circles, each of them of the form (3.10). In that case, if there exists at least one index $i \in [\![1, N]\!]$ such that $b_i - a_i < L_i$, then there exist solutions not converging to 0 as $t \to +\infty$, even if there is damping, cf. Example 3.7.

Remark 3.12. Equation (3.12) is satisfied when M is *left stochastic*, i.e., $m_{ij} \geq 0$ for every $i, j \in [\![1, N]\!]$ and $\sum_{i=1}^{N} m_{ij} = 1$ for every $j \in [\![1, N]\!]$. Note that left stochasticity of M is equivalent for the undamped system (3.3) to the preservation of $\sum_{i=1}^{N} \int_0^{L_i} u_i(t, x) dx$ and monotonicity of the solutions with respect to the initial conditions.

3.3 Explicit solution

This section provides a general formula for the explicit solution of (3.5). We first prove our formula in Section 3.3.1 in the simpler case of the undamped system (3.3), before turning to the general case in Section 3.3.2. The coefficients appearing in the formula will be characterized in Section 3.3.3.

3.3.1 The undamped system

Remark that, in order to obtain an explicit formula for $u_i(t, x)$ for $i \in [\![1, N]\!]$, $t \geq 0$ and $x \in [0, L_i]$, it suffices to obtain a formula for $u_i(t, 0)$ for $i \in [\![1, N]\!]$ and $t \geq 0$. Indeed, it is immediate to derive the following.

Lemma 3.13. *Let* $(u_{1,0}, \dots, u_{N,0}) \in D(A)$ *and let* $(u_1, \dots, u_N) \in \mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ *be the corresponding solution of* (3.3)*. Then, for every* $i \in [\![1, N]\!]$, $t \geq 0$, *and* $x \in [0, L_i]$, *we have*

$$u_i(t, x) = \begin{cases} u_{i,0}(x - t), & \text{if } 0 \leq t \leq x, \\ u_i(t - x, 0), & \text{if } t \geq x. \end{cases} \tag{3.13}$$

In order to express $u_i(t, 0)$ in terms of the initial condition $(u_{1,0}, \dots, u_{N,0}) \in D(A)$, we need to introduce some notation.

Definition 3.14.

(a) We define $\mathfrak{N} = \mathbb{N}^N$ and, for $i \in [\![1, N]\!]$, $\mathfrak{N}_i = \mathbb{N}^{i-1} \times \{0\} \times \mathbb{N}^{N-i}$.

(b) We write $\mathbf{0} = (0, 0, \dots, 0) \in \mathfrak{N}$ and, for every $j \in [\![1, N]\!]$ and $\mathfrak{n} = (n_1, \dots, n_N) \in \mathfrak{N}$, $\mathbf{1}_j = (\delta_{ij})_{i=1,\dots,N} \in \mathfrak{N}$ and $\hat{\mathfrak{n}}_j = (n_1, n_2, \dots, n_{j-1}, 0, n_{j+1}, \dots, n_N) = \mathfrak{n} - n_j \mathbf{1}_j \in \mathfrak{N}_j$.

(c) We define the function $L : \mathfrak{N} \to \mathbb{R}_+$ by

$$L(n_1, \dots, n_N) = \sum_{i=1}^{N} n_i L_i.$$

With these notations, the general formula for the solutions of (3.3) can be written as follows.

Theorem 3.15. *Let* $(u_{1,0}, \dots, u_{N,0}) \in D(A)$*. The corresponding solution* $(u_1, \dots, u_N)$ *of* (3.3) *is given by*

$$u_i(t, x) = \begin{cases} u_{i,0}(x - t), & \text{if } 0 \leq t \leq x, \\ u_i(t - x, 0), & \text{if } t \geq x, \end{cases} \tag{3.14}$$

with

$$u_i(t, 0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq t}} \beta^{(i)}_{j, \mathfrak{n} + \left\lfloor \frac{t - L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j} u_{j,0}\left(L_j - \{t - L(\mathfrak{n})\}_{L_j}\right), \tag{3.15}$$

and where the coefficients $\beta_{j,\mathfrak{n}}^{(i)}$ *are defined by the relations*

$$\beta_{j,\mathbf{0}}^{(i)} = m_{ij}, \qquad i, j \in [\![1, N]\!], \tag{3.16a}$$

and

$$\beta_{j,\mathfrak{n}}^{(i)} = \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} m_{kj} \beta_{k,\mathfrak{n}-\mathbf{1}_k}^{(i)}, \qquad i, j \in [\![1, N]\!],\ \mathfrak{n} \in \mathfrak{N} \setminus \{\mathbf{0}\}. \tag{3.16b}$$

The above result follows by iterating Equations (3.13) together with Equation (3.2). Indeed, using the notations of the theorem, one has for $i \in [\![1, N]\!]$,

$$u_i(t, 0) = \sum_{j=1}^{N} m_{ij} u_j(t, L_j). \tag{3.17}$$

According to Equation (3.13), each $u_j(t, L_j)$ is either equal to $u_{j,0}(L_j - t)$ or $u_j(t - L_j, 0)$, according to whether $t \leq L_j$ or not. In the latter case, we express $u_j(t - L_j, 0)$ by using Equation (3.17) and we repeat the procedure a finite number of times until obtaining $u_i(t, 0)$ as a linear combination involving only evaluations of the initial condition at finitely many points on the circles. This yields Equation (3.15) with an explicit expression of both the coefficients of this linear combination and the points on the circles.

The complete proof of Theorem 3.15 is provided in Appendix 3.C and consists in verifying that the explicit formula given in the above statement is indeed the solution of (3.3).

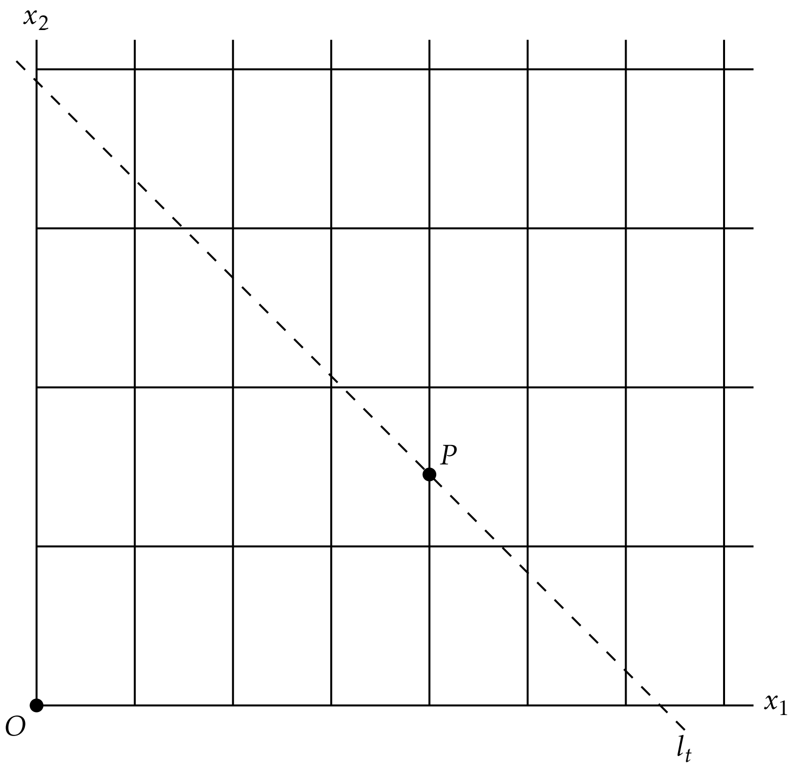


Figure 3.2: Geometric construction for the explicit formula for the solution of (3.3) in the case $N = 2$.

We next provide with Figure 3.2 a geometric interpretation of (3.15) in the case $N = 2$. The point O is identified with the origin of the plane $(x_1,\ x_2)$ and the horizontal (resp. vertical) segments in the grid represented in Figure 3.2 correspond to identical copies of the circle C_1 (resp. the circle C_2). The intersection of the dashed line $l_t : \ x_1 + x_2 = t$ and the grid exactly represents the set of points of the circles where the initial condition $(u_{1,0}, u_{2,0})$ is evaluated in Equation (3.15). Note that the coefficients in Equation (3.15) appearing in front of the evaluation of the initial condition at P can be expressed as a sum of products of the m_{ij}'s, each product corresponding to a path on the grid between P and O.

3.3.2 Formula for the explicit solution in the general case

We first notice that, as in Lemma 3.13, it suffices to study $u_i(t,0)$ for every $t \geq 0$ and $i \in [\![1,N]\!]$ in order to obtain the whole solution $(u_1(t),\dots,u_N(t))$. Recall that by convention we have set $\alpha_i \equiv 1$ and $a_i = b_i$ (and thus $\chi_i = 0$ almost everywhere) for $i \in [\![N_d+1,N]\!]$.

Proposition 3.16. *Let $(u_{1,0},\dots,u_{N,0}) \in D(A)$. Then the corresponding solution $(u_1,\dots,u_N)$ of (3.5) satisfies, for $i \in [\![1,N]\!]$,*

$$u_i(t,x) = \begin{cases} u_{i,0}(x-t)\exp\left(-\int_{[0,t]\cap[t-x+a_i,t-x+b_i]} \alpha_i(s)ds\right), & \text{if } 0 \leq t \leq x, \\ u_i(t-x,0)\exp\left(-\int_{[0,t]\cap[t-x+a_i,t-x+b_i]} \alpha_i(s)ds\right), & \text{if } t \geq x. \end{cases} \tag{3.18}$$

Proof. Let $i \in [\![1,N]\!]$. Equation (3.18) is obtained by integrating the differential equation

$$\frac{d}{ds} u_i(t+s,x+s) = -\alpha_i(t+s)\chi_i(x+s)u_i(t+s,x+s),$$

on the interval $[-t,0]$ if $t \leq x$ and on $[-x,0]$ if $t \geq x$. ∎

Thanks to the fact that all the exponential decays appearing in (3.18) are upper bounded by 1, one obtains trivially the following corollary.

Corollary 3.17. *If $(u_1,\dots,u_N)$ is the solution of (3.5) with an initial condition $(u_{1,0},\dots,u_{N,0})$, then, for $i \in [\![1,N_d]\!]$, u_i satisfies the estimate*

$$|u_i(t,x)| \leq \begin{cases} \left|u_{i,0}(x-t)\right|, & \text{if } 0 \leq t \leq x, \\ |u_i(t-x,0)|, & \text{if } t \geq x. \end{cases}$$

For every $p \in [1,+\infty]$, $i \in [\![1,N]\!]$, and $t \geq L_i$, we have

$$\|u_i(t,\cdot)\|_{L^p(0,L_i)} \leq \|u_i(\cdot,0)\|_{L^p(t-L_i,t)},$$

with equality if $i \in [\![N_d+1,N]\!]$.

This corollary allows us to replace the spatial L^p-norm of u_i at a given time t by its L^p-norm in a time interval of length L_i at the fixed position $x = 0$.

We can now write the explicit formula for the solutions of (3.5) using the notations from Definition 3.14. The proof follows the same steps as that of Theorem 3.15.

Theorem 3.18. *Let $(u_{1,0},\dots,u_{N,0}) \in D(A)$. The corresponding solution $(u_1,\dots,u_N)$ of (3.5) is given by (3.18), where $u_i(t,0)$ is given for $t \geq 0$ by*

$$u_i(t,0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j \\ L(\mathfrak{n})\leq t}} \vartheta^{(i)}_{j,\mathfrak{n}+\left\lfloor \frac{t-L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j, L_j-\{t-L(\mathfrak{n})\}_{L_j}, t}\, u_{j,0}\left(L_j - \{t-L(\mathfrak{n})\}_{L_j}\right) \tag{3.19}$$

and the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ *are defined for* $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0,L_j]$ *and* $t \in \mathbb{R}$ *by*

$$\vartheta^{(i)}_{j,\mathfrak{n},x,t} = \varepsilon_{j,\mathfrak{n},x,t}\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}, \tag{3.20a}$$

with

$$\varepsilon_{j,\mathfrak{n},x,t} = \exp\left(-\int_{I_{j,\mathfrak{n},x,t}} \alpha_j(s)ds\right), \tag{3.20b}$$

where $I_{j,\mathfrak{n},x,t} = [t - L(\mathfrak{n}) - L_j + \max(x,a_j), t - L(\mathfrak{n}) - L_j + b_j]$, *and*

$$\vartheta^{(i)}_{j,\mathbf{0},L_j,t} = m_{ij}, \tag{3.20c}$$

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \sum_{\substack{k=1\\ n_k \geq 1}}^{N} m_{kj}\vartheta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k,0,t}. \tag{3.20d}$$

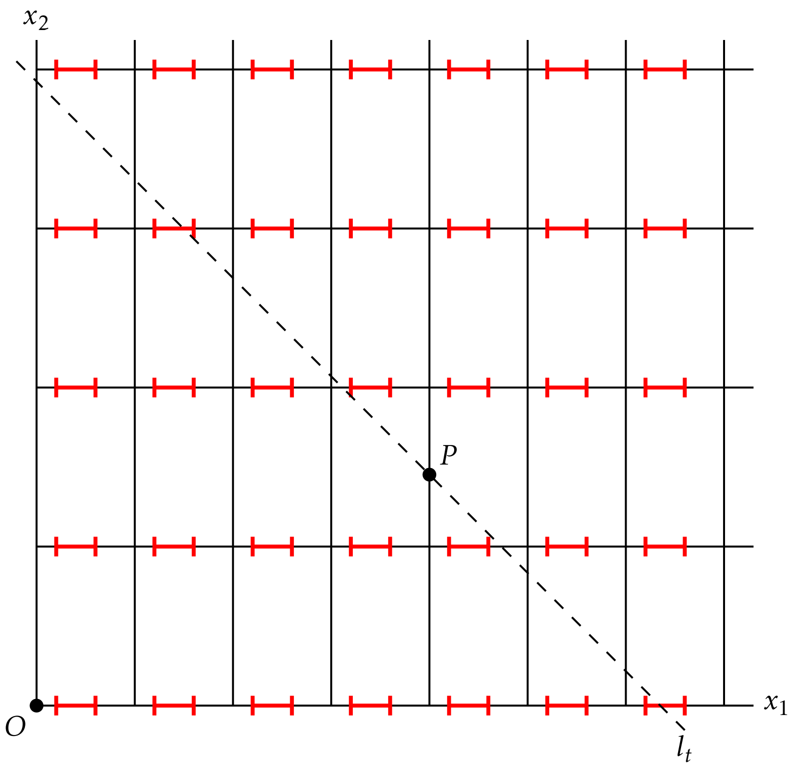


Figure 3.3: Geometric interpretation of the explicit formula (3.19) for the solution of (3.5) in the case $N = 2$.

Let us provide a geometrical interpretation of the above theorem in the case $N = 2$, $N_d = 1$ with damping on the circle C_1. With respect to Figure 3.2, Figure 3.3 now includes segments corresponding to the intervals $[a_1, b_1]$ in C_1, on which the solution is damped. Similarly to (3.15), the new explicit formula (3.19) expresses $u_i(t,0)$ as a linear combination involving only evaluations of the initial condition at finitely many points on the circles. The coefficients, which in (3.15) were sum of products of the m_{ij}'s, each product corresponding to a path between P and O, have now an analogous expression, with the following modification: each factor of the original product is multiplied by an additional term of the type $\varepsilon_{j,\mathfrak{n},x,t}$, which takes into account the effect of the damping along the path.

Remark 3.19. The explicit formula for the solution of the undamped equation (3.3) given in Theorem 3.15 can be obtained as a particular case of Theorem 3.18 by setting $\alpha_j \equiv 0$ for every $j \in [\![1, N_d]\!]$. Similarly, we can obtain the explicit formula for the solution of (3.4) as a particular case of Theorem 3.18 by setting $\alpha_j \equiv 1$ for every $j \in [\![1, N_d]\!]$, yielding $\varepsilon_{j,\mathfrak{n},x,t} = e^{-\operatorname{meas}([a_j,b_j]\cap[x,b_j])}$.

Remark 3.20. In the general case $(u_{1,0}, \dots, u_{N,0}) \in \mathsf{X}_p$, the mild solution $(u_1, \dots, u_N)$ of (3.5) can still be characterized by (3.18) and (3.19) (yielding an equality in X_p for every $t \geq 0$). This follows by a simple density argument of $D(A)$ in X_p.

3.3.3 Recursive formula for the coefficients

We now wish to determine a recursive formula with K steps for the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ appearing in the expression of the explicit solution (3.19). For $v \in [\![1, N]\!]^K$ and $k \in [\![1, N]\!]$, we denote

$$\varphi_{k,K}(v) = \sum_{s=1}^{K} \delta_{k v_s} = \#\{s \in [\![1, K]\!] \mid v_s = k\},$$

and, for $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} \geq K$, we introduce the set

$$\Phi_K(\mathfrak{n}) = \{v \in [\![1, N]\!]^K \mid n_j \geq \varphi_{j,K}(v) \text{ for all } j \in [\![1, N]\!]\}.$$

Then we have the following result.

Proposition 3.21. *Let $K \in \mathbb{N}^*$ and suppose that $\mathfrak{n} \in \mathfrak{N}$ is such that $|\mathfrak{n}|_{\ell^1} \geq K$. Then, for every $i, j \in [\![1, N]\!]$ and $t \in \mathbb{R}$, we have*

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \sum_{v \in \Phi_K(\mathfrak{n})} \left[\left(m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}}\right)\left(\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t}\right) \vartheta^{(i)}_{v_K, \mathfrak{n} - \sum_{s=1}^{K} \mathbf{1}_{v_s}, L_{v_K}, t}\right]. \tag{3.21}$$

Proof. The proof is done by induction on K. If $K = 1$, we have

$$\Phi_1(\mathfrak{n}) = \{v \in [\![1, N]\!] \mid n_j \geq \delta_{jv} \text{ for all } j \in [\![1, N]\!]\} = \{v \in [\![1, N]\!] \mid n_v \geq 1\},$$

and so, by (3.20a) and (3.20d),

$$\sum_{v \in \Phi_1(\mathfrak{n})} \left[m_{vj} \varepsilon_{v, \mathfrak{n} - \mathbf{1}_v, 0, t} \vartheta^{(i)}_{v, \mathfrak{n} - \mathbf{1}_v, L_v, t}\right] = \sum_{\substack{v=1 \\ n_v \geq 1}}^{N} m_{vj} \varepsilon_{v, \mathfrak{n} - \mathbf{1}_v, 0, t} \vartheta^{(i)}_{v, \mathfrak{n} - \mathbf{1}_v, L_v, t} = \vartheta^{(i)}_{j,\mathfrak{n},L_j,t}.$$

Suppose now that $K \in \mathbb{N}^*$ with $K \leq |\mathfrak{n}|_{\ell^1}$ and (3.21) holds true for $K - 1$. Then we have, by (3.20d),

$$\begin{aligned}
\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} &= \sum_{v' \in \Phi_{K-1}(\mathfrak{n})} \left[\left(m_{v'_1 j} \prod_{s=2}^{K-1} m_{v'_s v'_{s-1}}\right)\left(\prod_{s=1}^{K-1} \varepsilon_{v'_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v'_r}, 0, t}\right) \vartheta^{(i)}_{v'_{K-1}, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s}, L_{v'_{K-1}}, t}\right] \\
&= \sum_{v' \in \Phi_{K-1}(\mathfrak{n})} \left[\left(m_{v'_1 j} \prod_{s=2}^{K-1} m_{v'_s v'_{s-1}}\right)\left(\prod_{s=1}^{K-1} \varepsilon_{v'_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v'_r}, 0, t}\right)\right. \\
&\qquad \left.\left(\sum_{\substack{k=1 \\ n_k > \varphi_{k,K-1}(v')}}^{N} m_{k v'_{K-1}} \varepsilon_{k, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s} - \mathbf{1}_k, 0, t} \vartheta^{(i)}_{k, \mathfrak{n} - \sum_{s=1}^{K-1} \mathbf{1}_{v'_s} - \mathbf{1}_k, L_k, t}\right)\right] \\
&= \sum_{v \in \Phi_K(\mathfrak{n})} \left[\left(m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}}\right)\left(\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t}\right) \vartheta^{(i)}_{v_K, \mathfrak{n} - \sum_{s=1}^{K} \mathbf{1}_{v_s}, L_{v_K}, t}\right],
\end{aligned}$$

where we take $v = (v', v_N = k) = (v'_1, \dots, v'_{N-1}, k)$ and notice that $\varphi_{j,K}(v', k) = \varphi_{j,K-1}(v') + \delta_{jk}$ for every $j \in [\![1, N]\!]$, so that

$$\begin{aligned}&\{(v', k) \in \Phi_{K-1}(\mathfrak{n}) \times [\![1, N]\!] \mid n_k > \varphi_{k,K-1}(v')\} \\ &\quad = \{(v', k) \in [\![1, N]\!]^{K-1} \times [\![1, N]\!] \mid n_j \geq \varphi_{j,K-1}(v') \text{ for all } j \in [\![1, N]\!] \text{ and } n_k > \varphi_{k,K-1}(v')\} \\ &\qquad\qquad = \{v \in [\![1, N]\!]^K \mid n_j \geq \varphi_{j,K}(v) \text{ for all } j \in [\![1, N]\!]\} = \Phi_K(\mathfrak{n}).\end{aligned}$$

This proves (3.21) by induction. ■

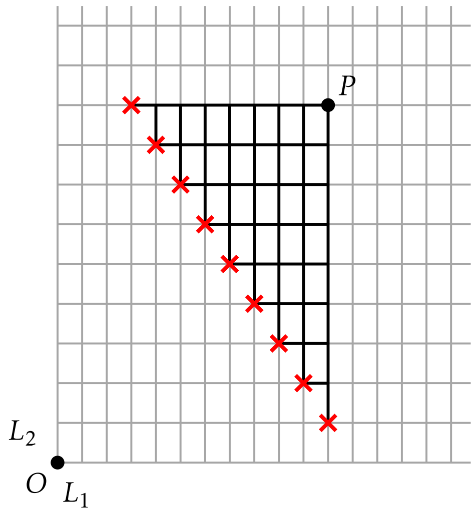


Figure 3.4: Here $N = 2$, $\mathfrak{n} = (11, 9)$ and $K = 8$.

Remark 3.22. In terms of the construction of Figure 3.3, each $v \in \Phi_K(\mathfrak{n})$ corresponds to a path between P and a point represented by a cross in Figure 3.4. The term

$$\prod_{s=1}^{K} \varepsilon_{v_s, \mathfrak{n} - \sum_{r=1}^{s} \mathbf{1}_{v_r}, 0, t}$$

in (3.21) represents the total decay along this path.

Remark 3.23. Under Hypothesis 3.11, the terms $m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}}$ in (3.21) satisfy

$$\sum_{v \in \Phi_K(\mathfrak{n})} \left| m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}} \right| \leq 1. \tag{3.22}$$

Indeed, we have

$$\sum_{v \in \Phi_K(\mathfrak{n})} \left| m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}} \right| \leq \sum_{v \in [\![1,N]\!]^K} \left| m_{v_1 j} \prod_{s=2}^{K} m_{v_s v_{s-1}} \right| = \sum_{v_1=1}^{N} \cdots \sum_{v_K=1}^{N} \left| m_{v_K v_{K-1}} \right| \cdots \left| m_{v_1 j} \right| \leq 1,$$

by applying iteratively (3.12). We also remark that Proposition 3.21 implies that the coefficient $\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}$ belongs to the convex hull of the points $\pm\vartheta^{(i)}_{v_K, \mathfrak{n} - \sum_{s=1}^{K} \mathbf{1}_{v_s}, L_{v_K}, t}$, for v in $\Phi_K(\mathfrak{n})$.

3.4 Proof of the main result

We now study the asymptotic behavior of the solutions of (3.5) with persistent damping, through their explicit formula obtained in Section 3.3. We first show that, in order to obtain the exponential stability of the solutions of (3.5), it suffices to obtain the exponential convergence as $|\mathfrak{n}|_{\ell^1} \to +\infty$ of the coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$ of the explicit formula (3.19).

3.4.1 Convergence of the coefficients implies convergence of the solution

The section is devoted to the proof of the following result.

Proposition 3.24. *Let $\mathcal{F} \subset L^\infty(\mathbb{R}, [0,1])$. Suppose that there exist constants $C_0, \gamma_0 > 0$ such that, for every $\alpha_k \in \mathcal{F}$, $k \in [\![1, N_d]\!]$, we have*

$$\left|\vartheta_{j,\mathfrak{n},x,t}^{(i)}\right| \leq C_0 e^{-\gamma_0 |\mathfrak{n}|_{\ell^1}}, \qquad \forall i,j \in [\![1,N]\!],\ \forall \mathfrak{n} \in \mathfrak{N},\ \forall x \in [0, L_j],\ \forall t \in \mathbb{R}.$$

Then there exist constants $C, \gamma > 0$ such that, for every $p \in [1, +\infty)$ and every initial condition $z_0 \in \mathsf{X}_p$, the corresponding solution z of (3.5) satisfies

$$\|z(t)\|_{\mathsf{X}_p} \leq C e^{-\gamma t} \|z_0\|_{\mathsf{X}_p}, \qquad \forall t \in \mathbb{R}_+. \tag{3.23}$$

Remark 3.25. The conclusion (3.23) of Proposition 3.24 can be written, in terms of the evolution family $\{T(t,s)\}_{t \geq s \geq 0}$ associated with (3.5), as

$$\|T(t,0)\|_{\mathcal{L}(\mathsf{X}_p)} \leq C e^{-\gamma t}, \qquad \forall t \in \mathbb{R}_+.$$

When the class $\mathcal{F}$ is invariant by time-translation (e.g., for $\mathcal{F} = \mathcal{G}(T, \mu)$), this is actually equivalent to

$$\|T(t,s)\|_{\mathcal{L}(\mathsf{X}_p)} \leq C e^{-\gamma(t-s)}, \qquad \forall t,s \in \mathbb{R}_+ \text{ with } t \geq s.$$

Proof. Take $z_0 = (u_{1,0}, \dots, u_{N,0}) \in \mathsf{X}_p$ and denote by $z(t) = (u_1(t), \dots, u_N(t))$ the corresponding solution of (3.5). By Corollary 3.17, Theorem 3.18, and Remark 3.20, we have, for $t \geq L_{\max}$,

$$\|z(t)\|_{\mathsf{X}_p}^p = \sum_{i=1}^N \|u_i(t)\|_{L^p(0,L_i)}^p \leq \sum_{i=1}^N \|u_i(\cdot, 0)\|_{L^p(t-L_i,t)}^p, \tag{3.24}$$

with $u_i(t,0)$ given by (3.19). Denoting $Y_j(t) = \#\{\mathfrak{n} \in \mathfrak{N}_j \mid L(\mathfrak{n}) \leq t\}$, we have

$$\begin{aligned}
\|u_i(\cdot,0)\|_{L^p(t-L_i,t)}^p &= \int_{t-L_i}^t |u_i(s,0)|^p \, ds \\
&\leq N^{p-1} \sum_{j=1}^N \int_{t-L_i}^t \left| \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq s}} \vartheta^{(i)}_{j, \mathfrak{n} + \left\lfloor \frac{s - L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j, L_j - \{s - L(\mathfrak{n})\}_{L_j}, s} u_{j,0}(L_j - \{s - L(\mathfrak{n})\}_{L_j}) \right|^p ds \\
&\leq N^{p-1} \sum_{j=1}^N \int_{t-L_i}^t Y_j(s)^{p-1} \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq s}} \left| \vartheta^{(i)}_{j, \mathfrak{n} + \left\lfloor \frac{s - L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j, L_j - \{s - L(\mathfrak{n})\}_{L_j}, s} u_{j,0}(L_j - \{s - L(\mathfrak{n})\}_{L_j}) \right|^p ds \\
&\leq N^{p-1} C_0^p \sum_{j=1}^N Y_j(t)^{p-1} \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq t}} \int_{t-L_i}^t e^{-p\gamma_0 \left(|\mathfrak{n}|_{\ell^1} + \left\lfloor \frac{s - L(\mathfrak{n})}{L_j} \right\rfloor\right)} \left| u_{j,0}(L_j - \{s - L(\mathfrak{n})\}_{L_j}) \right|^p ds
\end{aligned}$$

$$\leq N^{p-1} C_0^p e^{2p\gamma_0} e^{-\frac{p\gamma_0}{L_{\max}} t} \sum_{j=1}^{N} Y_j(t)^{p-1} \sum_{\substack{\mathfrak{n}\in\mathfrak{N}_j \\ L(\mathfrak{n})\leq t}} \int_{t-L_{\max}}^{t} \left| u_{j,0}(L_j - \{s - L(\mathfrak{n})\}_{L_j}) \right|^p ds, \quad (3.25)$$

where we use that

$$|\mathfrak{n}|_{\ell^1} + \left\lfloor \frac{s-L(\mathfrak{n})}{L_j} \right\rfloor = \sum_{k=1}^{N} n_k + \left\lfloor \frac{s-L(\mathfrak{n})}{L_j} \right\rfloor \geq \frac{L(\mathfrak{n})}{L_{\max}} + \left\lfloor \frac{s-L(\mathfrak{n})}{L_{\max}} \right\rfloor \geq \frac{s}{L_{\max}} - 1 \geq \frac{t}{L_{\max}} - 2,$$

for $\mathfrak{n} \in \mathfrak{N}_j$ with $L(\mathfrak{n}) \leq t$ and $s \in [t - L_i, t]$.

According to its definition, $Y_j(t)$ can be upper bounded as follows

$$\begin{aligned} Y_j(t) &\leq \#\{\mathfrak{n} \in \mathfrak{N}_j \mid n_i L_i \leq t \text{ for all } i \in [\![1,N]\!] \setminus \{j\}\} \\ &= \#\left(\left[\!\!\left[0, \frac{t}{L_1}\right]\!\!\right] \times \cdots \times \left[\!\!\left[0, \frac{t}{L_{j-1}}\right]\!\!\right] \times \{0\} \times \left[\!\!\left[0, \frac{t}{L_{j+1}}\right]\!\!\right] \times \cdots \times \left[\!\!\left[0, \frac{t}{L_N}\right]\!\!\right]\right) \leq \left(\frac{t}{L_{\min}} + 1\right)^{N-1}. \end{aligned} \quad (3.26)$$

We next estimate $\int_{t-L_{\max}}^{t} \left| u_{j,0}(L_j - \{s-L(\mathfrak{n})\}_{L_j}) \right|^p ds$ with $j \in [\![1,N]\!]$. Notice that $[t - L_{\max}, t) \subset \bigcup_{k=k_{\min}}^{k_{\max}} \left[L(\mathfrak{n}) + kL_j, L(\mathfrak{n}) + (k+1)L_j\right)$ with

$$k_{\min} = \max\{k \in \mathbb{Z} \mid L(\mathfrak{n}) + kL_j \leq t - L_{\max}\}, \quad k_{\max} = \min\{k \in \mathbb{Z} \mid L(\mathfrak{n}) + (k+1)L_j \geq t\}.$$

We deduce that

$$\begin{aligned} \int_{t-L_{\max}}^{t} \left| u_{j,0}(L_j - \{s-L(\mathfrak{n})\}_{L_j}) \right|^p ds &\leq \sum_{k=k_{\min}}^{k_{\max}} \int_0^{L_j} \left| u_{j,0}(\sigma) \right|^p d\sigma = (k_{\max} - k_{\min} + 1) \left\| u_{j,0} \right\|_{L^p(0,L_j)}^p \\ &\leq \left(\frac{L_{\max}}{L_j} + 2\right) \left\| u_{j,0} \right\|_{L^p(0,L_j)}^p. \end{aligned} \quad (3.27)$$

Inserting (3.26) and (3.27) into (3.25) finally gives (3.23) thanks to (3.24). Notice that the coefficients γ and C can be chosen to be independent of p. ■

Remark 3.26. Even though the well-posedness of (3.7) was considered in Section 3.2.1 only for $p \in [1, +\infty)$, we extend it here below to the case $p = +\infty$ and we verify that Proposition 3.24 still holds true in this case.

First set $\mathsf{X}_\infty = \prod_{i=1}^{N} L^\infty(0, L_i)$ with its usual norm $\|z\|_{\mathsf{X}_\infty} = \max_{i\in[\![1,N]\!]} \|u_i\|_{L^\infty(0,L_i)}$ for $z = (u_1, \ldots, u_N) \in \mathsf{X}_\infty$. Fix $z_0 \in \mathsf{X}_\infty$. Since $z_0 \in \mathsf{X}_p$ for every $p \in [1, +\infty)$, then (3.7) admits a unique mild solution $z(t) = T(t,0)z_0$ in $\bigcap_{p\in[1,+\infty)} \mathcal{C}^0(\mathbb{R}_+, \mathsf{X}_p)$ with initial condition $z(0) = z_0$. As noticed in Remark 3.20, $z(t)$ is characterized as an element of X_p by equations (3.18) and (3.19). Hence $z(t) \in \mathsf{X}_\infty$ for every $t \geq 0$. We can thus refer to $z(\cdot)$ as the solution of the Cauchy problem (3.7) in X_∞.

Suppose now that the hypotheses of Proposition 3.24 are satisfied and let $C, \gamma > 0$ be as in its statement. By (3.23), we have

$$\|z(t)\|_{\mathsf{X}_p} \leq Ce^{-\gamma t} \|z_0\|_{\mathsf{X}_p} \leq C \left(\sum_{i=1}^{N} L_i\right)^{1/p} e^{-\gamma t} \|z_0\|_{\mathsf{X}_\infty}.$$

Since $\left\|y\right\|_{\mathsf{X}_\infty} = \lim_{p\to+\infty} \left\|y\right\|_{\mathsf{X}_p}$ for every $y \in \mathsf{X}_\infty$, we conclude that

$$\|z(t)\|_{\mathsf{X}_\infty} \leq Ce^{-\gamma t} \|z_0\|_{\mathsf{X}_\infty}.$$

3.4.2 Preliminary estimates of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$

In this section we establish estimates on the growth of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ based on combinatorial arguments.

Proposition 3.27. *For every $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R}, [0, 1])$, $k \in [\![1, N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq |M|_{\ell^1}^{|\mathfrak{n}|_{\ell^1}+1}. \tag{3.28}$$

Proof. We show (3.28) by induction on $|\mathfrak{n}|_{\ell^1}$. For every $i, j \in [\![1, N]\!]$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R}, [0, 1])$, $k \in [\![1, N_d]\!]$, we have, by (3.20),

$$\left|\vartheta^{(i)}_{j,\mathbf{0},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathbf{0},L_j,t}\right| = \left|m_{ij}\right| \leq |M|_{\ell^1}.$$

If $R \in \mathbb{N}$ is such that (3.28) holds for every $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R}, [0, 1])$, $k \in [\![1, N_d]\!]$, then, for $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R + 1$, we have, by (3.20),

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \sum_{\substack{r=1\\ n_r\geq 1}}^{N} \left|m_{rj}\right| \left|\vartheta^{(i)}_{r,\mathfrak{n}-\mathbf{1}_r,L_r,t}\right| \leq \sum_{r=1}^{N} \left|m_{rj}\right| |M|_{\ell^1}^{R+1} \leq |M|_{\ell^1}^{R+2},$$

since $|M|_{\ell^1} = \max_{j\in[\![1,N]\!]} \sum_{r=1}^{N} \left|m_{rj}\right|$. The result thus follows by induction. ∎

As a consequence of Propositions 3.24 and 3.27 and Remark 3.26, we deduce at once the following corollary.

Corollary 3.28. *Suppose that $|M|_{\ell^1} < 1$. Then there exist $C, \gamma > 0$ such that, for every $p \in [1, +\infty]$ and every initial condition $z_0 \in \mathsf{X}_p$, the corresponding solution $z(t)$ of the undamped equation (3.3) satisfies*

$$\|z(t)\|_{\mathsf{X}_p} \leq C e^{-\gamma t} \|z_0\|_{\mathsf{X}_p}, \qquad \forall t \in \mathbb{R}_+.$$

Another trivial but important consequence of Proposition 3.27 is that, if $|M|_{\ell^1} \leq 1$, the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ are all bounded in absolute value by 1.

Corollary 3.29. *Suppose that $|M|_{\ell^1} \leq 1$. Then, for every $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in L^\infty(\mathbb{R}, [0, 1])$, $k \in [\![1, N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq 1. \tag{3.29}$$

Our second estimate on the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ is the following.

Lemma 3.30. *Suppose that Hypothesis 3.11 is satisfied. Then there exists $\nu \in (0, 1)$ such that, for every $i, j, k \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R}, [0, 1])$, $r \in [\![1, N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \binom{|\mathfrak{n}|_{\ell^1}}{n_k} \nu^{|\mathfrak{n}|_{\ell^1}}. \tag{3.30}$$

Proof. Up to a permutation in the set of indices, we can suppose, without loss of generality, that $k = N$. Let

$$\mu_N = \max_{j\in[\![1,N]\!]} \sum_{i=1}^{N-1} \left|m_{ij}\right|, \qquad \nu_N = \max_{j\in[\![1,N]\!]} \left|m_{Nj}\right|.$$

By Hypothesis 3.11, we have both $\sum_{i=1}^{N-1} \left|m_{ij}\right| < 1$ and $\left|m_{Nj}\right| < 1$. Hence $\mu_N, \nu_N \in (0,1)$.

We prove by induction on $|\mathfrak{n}|_{\ell^1}$ that

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \binom{|\mathfrak{n}|_{\ell^1}}{n_k} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N}. \tag{3.31}$$

For every $i, j \in [\![1,N]\!]$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$, we have, by (3.20),

$$\left|\vartheta^{(i)}_{j,\mathbf{0},x,t}\right| \leq \left|\vartheta^{(i)}_{j,\mathbf{0},L_j,t}\right| = \left|m_{ij}\right| \leq 1,$$

so that (3.31) is satisfied for $\mathfrak{n} = \mathbf{0}$.

Suppose now that $R \in \mathbb{N}$ is such that (3.31) is satisfied for every $i, j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ with $|\mathfrak{n}|_{\ell^1} = R$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R},[0,1])$, $r \in [\![1,N_d]\!]$. If $\mathfrak{n} \in \mathfrak{N}$ is such that $|\mathfrak{n}|_{\ell^1} = R+1$, we have, by (3.20),

$$\begin{aligned}
\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| &\leq \left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \sum_{\substack{r=1\\ n_r\geq 1}}^{N-1} \left|m_{rj}\right| \left|\vartheta^{(i)}_{r,\mathfrak{n}-\mathbf{1}_r,L_r,t}\right| + \left|m_{Nj}\right| \left|\vartheta^{(i)}_{N,\mathfrak{n}-\mathbf{1}_N,L_N,t}\right| \\
&\leq \sum_{\substack{r=1\\ n_r\geq 1}}^{N-1} \left|m_{rj}\right| \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N-1} \nu_N^{n_N} + \left|m_{Nj}\right| \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N-1} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N-1} \\
&\leq \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N} + \binom{|\mathfrak{n}|_{\ell^1}-1}{n_N-1} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N} = \binom{|\mathfrak{n}|_{\ell^1}}{n_N} \mu_N^{|\mathfrak{n}|_{\ell^1}-n_N} \nu_N^{n_N},
\end{aligned}$$

with the convention that $\vartheta^{(i)}_{N,\mathfrak{n}-\mathbf{1}_N,L_N,t} = 0$ if $n_N = 0$. Hence (3.31) holds for $\mathfrak{n}$, which proves the result by induction. We conclude by taking $\nu = \max\{\mu_k, \nu_k \mid k \in [\![1,N]\!]\}$. ■

3.4.3 Exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_b(\rho)$

The proof of the exponential decay of the coefficients $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ as $|\mathfrak{n}|_{\ell^1} \to +\infty$, uniformly with respect to $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, is split into two cases. We first estimate $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ for $\mathfrak{n}$ in a subset $\mathfrak{N}_b(\rho)$ of $\mathfrak{N}$, namely when one of the components of $\mathfrak{n}$ is much smaller than the others. The parameter $\rho \in (0,1)$ is a measure of such a smallness and will be fixed later. For $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ does not result from the presence of the persistent damping but solely from combinatorial considerations. We then proceed in Section 3.4.4 to estimate $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_c(\rho) = \mathfrak{N} \setminus \mathfrak{N}_b(\rho)$, where the decay comes from the persistent damping in (3.5).

Definition 3.31. For $k \in [\![1,N]\!]$ and $\rho \in (0,1)$, we define

$$\mathfrak{N}_b(\rho,k) = \{\mathfrak{n} = (n_1,\dots,n_N) \in \mathfrak{N} \mid n_k \leq \rho |\mathfrak{n}|_{\ell^1}\},$$

$$\mathfrak{N}_b(\rho) = \bigcup_{k=1}^{N} \mathfrak{N}_b(\rho,k), \qquad \mathfrak{N}_c(\rho) = \mathfrak{N} \setminus \mathfrak{N}_b(\rho).$$

We now deduce from Lemma 3.30 the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_b(\rho)$.

Theorem 3.32. *Suppose that Hypothesis 3.11 is satisfied. There exist $\rho \in (0, 1/2)$ and constants $C, \gamma > 0$ such that, for every $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, $x \in [0, L_j]$, $t \in \mathbb{R}$, and $\alpha_r \in L^\infty(\mathbb{R}, [0,1])$, $r \in [\![1, N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}. \tag{3.32}$$

Proof. Let $\nu \in (0,1)$ be as in Lemma 3.30. According to Lemma 3.60 in Appendix 3.D, there exist $\rho \in (0, 1/2)$, $C, \gamma > 0$ such that for every $n \in \mathbb{N}$ and $k \in [\![0, \rho n]\!]$, we have $\binom{n}{k}\nu^n \leq Ce^{-\gamma n}$. Take $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, $x \in [0, L_j]$, and $\alpha_r \in L^\infty(\mathbb{R}, [0,1])$ for $r \in [\![1, N_d]\!]$. Since $\mathfrak{n} \in \mathfrak{N}_b(\rho)$, there exists $k \in [\![1, N]\!]$ such that $\mathfrak{n} \in \mathfrak{N}_b(\rho, k)$, i.e., $n_k \leq \rho|\mathfrak{n}|_{\ell^1}$. Then, by Lemmas 3.30 and 3.60, we have

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq \binom{|\mathfrak{n}|_{\ell^1}}{n_k}\nu^{|\mathfrak{n}|_{\ell^1}} \leq Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}. \qquad \blacksquare$$

Remark 3.33. The above estimate is actually sufficient to derive the conclusion of Theorem 3.1 when the damping is always active; indeed, in this case, one can easily deduce by an inductive argument using (3.20) that

$$\vartheta^{(i)}_{j,\mathfrak{n},L_j,t} = \beta^{(i)}_{j,\mathfrak{n}} e^{-n_1(b_1-a_1)} e^{-n_2(b_2-a_2)} \dots e^{-n_{N_d}(b_{N_d}-a_{N_d})}$$

and the exponential decay in $\mathfrak{N}_c(\rho)$ follows straightforwardly. Notice that in this case Hypothesis 3.10 is not necessary.

3.4.4 Exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_c(\rho)$

In this section, we establish the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in the set $\mathfrak{N}_c(\rho)$. The main difficulty in proving it lies in the fact that $\alpha_i(t)$ can be equal to zero for certain time intervals, so that the term $\varepsilon_{j,\mathfrak{n},x,t}$ defined by (3.20b) can be equal to 1. Recall that

$$\varepsilon_{j,\mathfrak{n},0,t} = e^{-\int_{t-L(\mathfrak{n})-L_j+a_j}^{t-L(\mathfrak{n})-L_j+b_j} \alpha_j(s)ds}. \tag{3.33}$$

Our goal consists in showing in Lemma 3.38 that $\varepsilon_{j,\mathfrak{n},0,t}$ is smaller than a certain value "often enough". The first step in this direction is the following lemma.

Lemma 3.34. *Let $T \geq \mu > 0$ and $j \in [\![1, N_d]\!]$. For $\rho > 0$ and $\alpha \in \mathcal{G}(T, \mu)$, define*

$$\mathcal{I}_{j,\rho,\alpha} = \left\{\tau \in \mathbb{R} \,\middle|\, \int_{\tau+a_j}^{\tau+b_j} \alpha(s)ds \geq \rho\right\}. \tag{3.34}$$

There exist $\rho_j > 0$ and $\ell_j > 0$, depending only on μ, T and $b_j - a_j$, such that, for every $t \in \mathbb{R}$ and $\alpha \in \mathcal{G}(T, \mu)$, $\mathcal{I}_{j,\rho_j,\alpha} \cap [t, t+T]$ contains an interval of length ℓ_j.

Proof. We set $\rho_j = \frac{\mu(b_j-a_j)}{2T}$, $\ell_j = \min\{\rho_j, T\}$. Take $\alpha \in \mathcal{G}(T, \mu)$ and define the function $A : \mathbb{R} \to \mathbb{R}$ by

$$A(\tau) = \int_{\tau+a_j}^{\tau+b_j} \alpha(s)ds.$$

Since $\alpha \in L^\infty(\mathbb{R}, [0,1])$, A is 1-Lipschitz continuous. We also have, for every $t \in \mathbb{R}$,

$$\int_t^{t+T} A(\tau)d\tau = \int_t^{t+T}\int_{a_j}^{b_j} \alpha(s+\tau)dsd\tau = \int_{a_j}^{b_j}\int_{s+t}^{s+t+T} \alpha(\tau)d\tau ds \geq \mu(b_j - a_j). \tag{3.35}$$

Take $t \in \mathbb{R}$. There exists $t_\star \in [t, t+T]$ such that $A(t_\star) \geq \frac{\mu(b_j - a_j)}{T} = 2\rho_j$, for otherwise (3.35) would not be satisfied. Since A is 1-Lipschitz continous, we have $A(\tau) \geq \rho_j$ for $\tau \in [t_\star - \rho_j, t_\star + \rho_j]$, and thus

$$\left[t_\star - \rho_j, t_\star + \rho_j\right] \cap [t, t+T] \subset \mathcal{I}_{j,\rho_j,\alpha} \cap [t, t+T].$$

But, since $t_\star \in [t, t+T]$, $[t_\star - \rho_j, t_\star + \rho_j] \cap [t, t+T]$ is an interval of length at least ℓ_j, which concludes the proof. ∎

Lemma 3.34 translates the persistence of excitation of α into a property on the integrals appearing in (3.33).

As remarked in Section 3.2.3, one cannot expect to obtain a general result concerning the exponential stability of (3.5) without taking into account the rationality of the ratios L_i/L_j. The following lemma uses the irrationality of L_i/L_j for certain $i, j \in [\![1, N]\!]$ to give a further step into the understanding of $\varepsilon_{j,\mathfrak{n},0,t}$.

Lemma 3.35. *Let $T \geq \mu > 0$ and let $\rho_j > 0$, $j \in [\![1, N_d]\!]$, be as in Lemma 3.34. There exists $K \in \mathbb{N}$ such that, for every $k_1 \in [\![1, N]\!]$, $k_2 \in [\![1, N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha \in \mathcal{G}(T, \mu)$, there exists $\mathfrak{r} \in \mathfrak{N}$ with $n_j \leq r_j \leq K + n_j$, $j \in \{k_1, k_2\}$, and $r_j = n_j$ for $j \in [\![1, N]\!] \setminus \{k_1, k_2\}$, such that*

$$t - L(\mathfrak{r}) \in \mathcal{I}_{k_2, \rho_{k_2}, \alpha}.$$

Proof. We shall prove the following simpler statement: for every $k_1 \in [\![1, N]\!]$ and $k_2 \in [\![1, N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, there exist $N_1 = N_1(k_1, k_2) \in \mathbb{N}$ and $N_2 = N_2(k_1, k_2) \in \mathbb{N}$ such that, for every $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha \in \mathcal{G}(T, \mu)$, there exist $\mathfrak{r} \in \mathfrak{N}$ with $n_{k_j} \leq r_{k_j} \leq N_j + n_{k_j}$, $j \in \{1, 2\}$, and $r_j = n_j$ for $j \in [\![1, N]\!] \setminus \{k_1, k_2\}$, such that

$$t - L(\mathfrak{r}) \in \mathcal{I}_{k_2, \rho_{k_2}, \alpha}.$$

From this result, one can easily obtain the statement of the lemma by taking

$$K = \max\left\{N_1(k_1, k_2),\ N_2(k_1, k_2) \;\middle|\; k_1 \in [\![1, N]\!],\ k_2 \in [\![1, N_d]\!] \text{ such that } \frac{L_{k_1}}{L_{k_2}} \notin \mathbb{Q}\right\}.$$

We decompose the argument into two steps.

Step 1. *Definition of the points x_j and y_j.*

Let $\rho_{k_2} > 0$ and $\ell_{k_2} > 0$ be obtained from μ, T and $b_{k_2} - a_{k_2}$ as in Lemma 3.34. Let $\kappa = 3\left\lceil T/\ell_{k_2}\right\rceil$ and set

$$x_j = \frac{j}{\kappa} T, \qquad j \in [\![0, \kappa]\!],$$

which satisfy $x_j - x_{j-1} = \frac{\kappa}{T} \leq \frac{\ell_{k_2}}{3}$ for $j \in [\![1, \kappa]\!]$. Hence, for every interval J of length ℓ_{k_2} contained in $[0, T]$, there exists $j \in [\![1, \kappa]\!]$ such that $x_{j-1}, x_j \in J$.

We now construct intermediate points between the x_j, $j \in [\![0, \kappa]\!]$. Since $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, the set

$$\{n_1 L_{k_1} + n_2 L_{k_2} \mid n_1, n_2 \in \mathbb{Z}\} \tag{3.36}$$

is dense in $\mathbb{R}$. Hence we can find $n_{1,j}, n_{2,j} \in \mathbb{Z}$, $j \in [\![1, \kappa]\!]$, such that the numbers $y_j = n_{1,j} L_{k_1} + n_{2,j} L_{k_2}$ satisfy

$$0 = x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_\kappa < x_\kappa = T. \tag{3.37}$$

As a consequence, for any interval J of length ℓ_{k_2} contained in $[0, T]$, there exists $j \in [\![1, \kappa]\!]$ such that $y_j \in J$.

Step 2. *Characterization of $\mathfrak{r} \in \mathfrak{N}$ and conclusion.*
Let $N_1^\star = \max\{|n_{1,1}|, \dots, |n_{1,\kappa}|\}$, $N_2^\star = \max\{|n_{2,1}|, \dots, |n_{2,\kappa}|\}$ and $N_1 = 2N_1^\star$, $N_2 = 2N_2^\star$. Take $t \in \mathbb{R}$ and $\mathfrak{n} \in \mathfrak{N}$. For $j \in [\![1, \kappa]\!]$, define $\mathfrak{r}_j = (r_{1,j}, \dots, r_{N,j}) \in \mathfrak{N}$ by

$$r_{k_1,j} = n_{k_1} + n_{1,j} + N_1^\star, \qquad r_{k_2,j} = n_{k_2} + n_{2,j} + N_2^\star,$$

and $r_{i,j} = n_i$ for $i \in [\![1, N]\!] \setminus \{k_1, k_2\}$; it is clear, by this definition, that $n_{k_i} \leq r_{k_i,j} \leq N_i + n_{k_i}$ for $i \in \{1, 2\}$ and $j \in [\![1, \kappa]\!]$. Set

$$z_j = t - L(\mathfrak{r}_j), \qquad j \in [\![1, \kappa]\!];$$

we thus have

$$z_j = t - n_{1,j} L_{k_1} - n_{2,j} L_{k_2} - Z^\star = t - Z^\star - y_j$$

with $Z^\star = L(\mathfrak{n}) + N_1^\star L_{k_1} + N_2^\star L_{k_2}$. Since, by construction, $y_j \in (0, T)$ for $j \in [\![1, \kappa]\!]$, we have $z_j \in [t - Z^\star - T, t - Z^\star]$.
Take $\alpha \in \mathcal{G}(T, \mu)$. By Lemma 3.34, $\mathcal{I}_{k_2, \rho_{k_2}, \alpha} \cap [t - Z^\star - T, t - Z^\star]$ contains an interval J of length ℓ_{k_2}. Consider the interval $J' = -J + t - Z^\star$, which is a subinterval of $[0, T]$ of length ℓ_{k_2}. By Step 1, there exists $j \in [\![1, \kappa]\!]$ such that $y_j \in J'$, and thus $z_j \in J \subset \mathcal{I}_{k_2, \rho_{k_2}, \alpha}$. Since $z_j = t - L(\mathfrak{r}_j)$, we obtain the desired result with $\mathfrak{r} = \mathfrak{r}_j$. ∎

Remark 3.36. The only instance in the proof of Lemma 3.35 where we use the fact that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$ is when we establish the existence of numbers y_j, $j = 1, \dots, \kappa$, of the form $y_j = n_{1,j} L_{k_1} + n_{2,j} L_{k_2}$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (3.37), which we do by using the density of the set (3.36). When $L_{k_1}/L_{k_2} \in \mathbb{Q}$ and we write $L_{k_1}/L_{k_2} = p/q$ for coprime $p, q \in \mathbb{N}^*$, the set given in (3.36) is, by Bézout's Lemma,

$$\left\{ L_{k_2} \frac{n_1 p + n_2 q}{q} \,\middle|\, n_1, n_2 \in \mathbb{Z} \right\} = \left\{ k \frac{L_{k_2}}{q} \,\middle|\, k \in \mathbb{Z} \right\}.$$

Hence the construction of $y_j = n_{1,j} L_{k_1} + n_{2,j} L_{k_2}$ with $n_{1,j}, n_{2,j} \in \mathbb{Z}$ satisfying (3.37) is still possible if $L_{k_2}/q < \kappa/T$, i.e., if $q > L_{k_2}\kappa/T$, and thus Lemma 3.35 still holds true if $L_{k_1}/L_{k_2} = p/q$ with coprime $p, q \in \mathbb{N}$ and q large enough.

Recalling that $\kappa = 3\lceil T/\ell_{k_2} \rceil$ with $\ell_{k_2} = \min\left\{\frac{\mu(b_{k_2} - a_{k_2})}{2T}, T\right\}$, we can even give a more explicit sufficient condition on q to still have Lemma 3.35: if

$$q \geq 3 L_{k_2} \left(\max\left\{ \frac{2T}{\mu(b_{k_2} - a_{k_2})}, \frac{1}{T} \right\} + \frac{1}{T} \right),$$

then one can easily check that $q > L_{k_2}\kappa/T$ and hence we are in the previous situation.

More explicitly, we can replace Hypothesis 3.10 by the following one.

Hypothesis 3.37. There exist $i \in [\![1, N]\!]$ and $j \in [\![1, N_d]\!]$ for which we have either $L_i/L_j \notin \mathbb{Q}$ or $L_i/L_j = p/q$ with coprime $p, q \in \mathbb{N}^*$ satisfying

$$q \geq 3 L_j \left(\max\left\{ \frac{2T}{\mu(b_j - a_j)}, \frac{1}{T} \right\} + \frac{1}{T} \right). \tag{3.38}$$

Notice that condition (3.38) only depends on the constants T, μ of the persistence of excitation condition, on the length $b_j - a_j$ of the damping interval j and on the length L_j.

As a consequence of the previous lemma we deduce the following property.

Lemma 3.38. *Let $T \geq \mu > 0$. There exist $\eta \in (0,1)$ and $K \in \mathbb{N}$ such that, for every $k_1 \in [\![1,N]\!]$, $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$, $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha_{k_2} \in \mathcal{G}(T,\mu)$, there exists $\mathfrak{r} \in \mathfrak{N}$ with $n_j \leq r_j \leq K + n_j$, $j \in \{k_1,k_2\}$, and $r_j = n_j$ for $j \in [\![1,N]\!] \setminus \{k_1,k_2\}$, such that*

$$\varepsilon_{k_2,\mathfrak{r},0,t} \leq \eta.$$

Proof. Take $\rho_j > 0$, $j \in [\![1,N_d]\!]$, as in Lemma 3.34 and $K \in \mathbb{N}$ as in Lemma 3.35. Define $\eta = \max_{j\in[\![1,N_d]\!]} e^{-\rho_j} \in (0,1)$. Take $k_1 \in [\![1,N]\!]$ and $k_2 \in [\![1,N_d]\!]$ with $L_{k_1}/L_{k_2} \notin \mathbb{Q}$. Let $t \in \mathbb{R}$, $\mathfrak{n} \in \mathfrak{N}$, and $\alpha_{k_2} \in \mathcal{G}(T,\mu)$. Applying Lemma 3.35 at $t - L_{k_2}$, we deduce the existence of $\mathfrak{r} \in \mathfrak{N}$ such that

$$t - L_{k_2} - L(\mathfrak{r}) \in \mathcal{I}_{k_2,\rho_{k_2},\alpha_{k_2}}.$$

By the definition (3.34) of $\mathcal{I}_{k_2,\rho_{k_2},\alpha_{k_2}}$, this means that

$$\int_{t-L(\mathfrak{r})-L_{k_2}+a_{k_2}}^{t-L(\mathfrak{r})-L_{k_2}+b_{k_2}} \alpha_{k_2}(s)ds \geq \rho_{k_2}.$$

By (3.33), we thus obtain that $\varepsilon_{k_2,\mathfrak{r},0,t} \leq e^{-\rho_{k_2}} \leq \eta$. ∎

Remark 3.39. Notice that the hypothesis that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$ is only used to apply Lemma 3.35, and thus, by Remark 3.36, Lemma 3.38 still holds true if k_1 and k_2 are chosen as in Hypothesis 3.37.

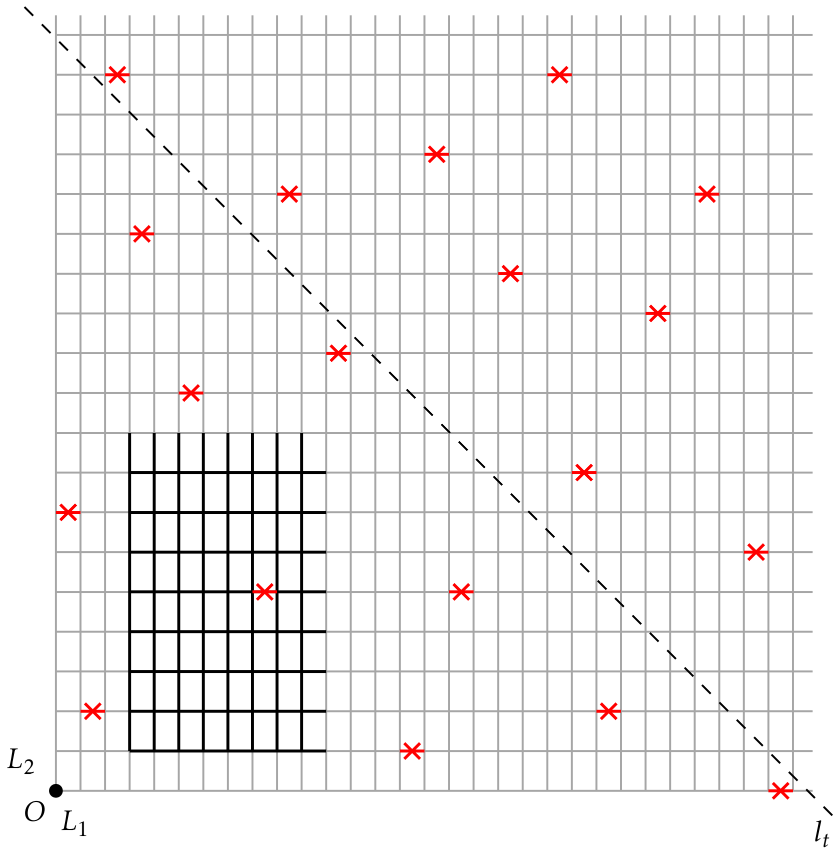


Figure 3.5: Interpretation of Lemma 3.38 in the case $N = 2$ and $N_d = 1$.

Remark 3.40. Figure 3.5 helps illustrating Lemma 3.38 in the case $N = 2$. Taking $N_d = 1$, the lemma states that there exists $K \in \mathbb{N}$ ($K = 7$ in the picture) such that every rectangle of size $(K+1)L_{k_1} \times (K+1)L_{k_2}$ that we place in the grid pictured in Figure 3.5 contains at least one horizontal segment (highlighted in the picture) where $\varepsilon_{k_2,\mathfrak{r},0,t} \leq \eta$. Let us remark that K and η do not depend on $\alpha_1 \in \mathcal{G}(T,\mu)$: hence the position of the highlighted segments may change if we change the persistently exciting signal, but we can guarantee that on every rectangle there exists at least one such segment.

We now apply Lemma 3.38 to obtain the following property, which is a preliminary step towards the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ in $\mathfrak{N}_c(\rho)$.

Lemma 3.41. *Suppose that Hypotheses 3.10 and 3.11 are satisfied. Let $T \geq \mu > 0$ and $\rho \in (0,1)$. Then there exist $\lambda \in (0,1)$ and $K \in \mathbb{N}^*$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i \in [\![1,N]\!]} n_i \geq K$, $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \lambda \max_{\substack{\mathfrak{p} \in \mathfrak{N} \,|\, \lvert\mathfrak{p}\rvert_{\ell^1} = K \\ p_r \leq n_r,\, \forall r \in [\![1,N]\!] \\ s \in [\![1,N]\!] \,|\, p_s > 0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

Proof. Let $\eta \in (0,1)$ and $K_0 \in \mathbb{N}^*$ be as in Lemma 3.38. Let $L_{\min} = \min_{i \in [\![1,N]\!]} L_i$. We take $K = 2K_0 + 1$.

Take $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i \in [\![1,N]\!]} n_i \geq K$ and let $k_1 \in [\![1,N]\!]$, $k_2 \in [\![1,N_d]\!]$ be such that $L_{k_1}/L_{k_2} \notin \mathbb{Q}$. Since $\mathfrak{n} \in \mathfrak{N}_c(\rho)$, one has $n_{k_i} > \rho \lvert\mathfrak{n}\rvert_{\ell^1}$ for $i \in \{1,2\}$. Take $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T,\mu)$ for $k \in [\![1,N_d]\!]$. Since $\lvert\mathfrak{n}\rvert_{\ell^1} \geq \min_{i \in [\![1,N]\!]} n_i \geq K$, we can apply Proposition 3.21 and deduce from (3.21) the estimate

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \Theta \sum_{v \in \Phi_K(\mathfrak{n})} \left[\left(\lvert m_{v_1 j}\rvert \prod_{s=2}^{K} \lvert m_{v_s v_{s-1}}\rvert\right)\left(\prod_{s=1}^{K} \varepsilon_{v_s,\mathfrak{n}-\sum_{r=1}^{s} \mathbf{1}_{v_r},0,t}\right)\right], \tag{3.39}$$

where

$$\Theta = \max_{\substack{\mathfrak{p} \in \mathfrak{N} \,|\, \lvert\mathfrak{p}\rvert_{\ell^1} = K \\ p_r \leq n_r,\, \forall r \in [\![1,N]\!] \\ s \in [\![1,N]\!] \,|\, p_s > 0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

Let us now apply Lemma 3.38 to the point $\mathfrak{n}' = \mathfrak{n} - K_0 \mathbf{1}_{k_1} - (K_0+1)\mathbf{1}_{k_2}$. Notice that $\mathfrak{n}' \in \mathfrak{N}$ since $n_{k_1}, n_{k_2} > K_0$. Hence there exists $\mathfrak{r} \in \mathfrak{N}$ with

$$\begin{aligned} n_{k_1} - K_0 \leq r_{k_1} &\leq n_{k_1}, \\ n_{k_2} - K_0 - 1 \leq r_{k_2} &\leq n_{k_2} - 1, \\ r_j = n_j \text{ for } j &\in [\![1,N]\!] \setminus \{k_1,k_2\}, \end{aligned}$$

such that $\varepsilon_{k_2,\mathfrak{r},0,t} \leq \eta$.

We next show that there exists $v_0 = (v_{0,1}, \dots, v_{0,K}) \in \Phi_K(\mathfrak{n})$ and $s_0 \in [\![1,K]\!]$ such that $v_{0,s_0} = k_2$ and $\mathfrak{n} - \sum_{r=1}^{s_0} \mathbf{1}_{v_{0,r}} = \mathfrak{r}$. For that purpose, take $v_0 \in [\![1,N]\!]^K$ with $v_{0,1} = v_{0,2} = \cdots = v_{0,n_{k_1}-r_{k_1}} = k_1$ and $v_{0,n_{k_1}-r_{k_1}+1} = v_{0,n_{k_1}-r_{k_1}+2} = \cdots = v_{0,K} = k_2$. Such a v_0 is well-defined in $\Phi_K(\mathfrak{n})$ since $0 \leq n_{k_1} - r_{k_1} \leq K_0 < K$. By construction, $\varphi_{k_1,K}(v_0) = n_{k_1} - r_{k_1} \leq n_{k_1}$, $\varphi_{k_2,K}(v_0) = K - (n_{k_1} - r_{k_1}) \leq K \leq n_{k_2}$ and $\varphi_{k,K}(v_0) = 0 \leq n_k$ for $k \in [\![1,N]\!] \setminus \{k_1,k_2\}$. Hence v_0 is in $\Phi_K(\mathfrak{n})$. Taking $s_0 = n_{k_1} - r_{k_1} + n_{k_2} - r_{k_2} \in [\![1,K]\!]$, we have $v_{0,s_0} = k_2$ and $\mathfrak{n} - \sum_{r=1}^{s_0} \mathbf{1}_{v_{0,r}} = \mathfrak{r}$.

Let $\delta = \min_{i,j\in[\![1,N]\!]} \left|m_{ij}\right| > 0$ and $\lambda = 1 - \delta^K(1-\eta)$. One clearly has that λ is in $(0,1)$, since $\eta, \delta \in (0,1)$. Using (3.22), we get from (3.39) that

$$\begin{aligned}
\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| &\leq \Theta\Bigg[\left(\left|m_{v_{0,1}j}\right|\prod_{s=2}^{K}\left|m_{v_{0,s}v_{0,s-1}}\right|\right)\left(\varepsilon_{v_{0,s_0},\mathfrak{n}-\sum_{r=1}^{s_0}\mathbf{1}_{v_{0,r}},0,t}\prod_{\substack{s=1\\ s\neq s_0}}^{K}\varepsilon_{v_{0,s},\mathfrak{n}-\sum_{r=1}^{s}\mathbf{1}_{v_{0,r}},0,t}\right)\\
&\quad+\sum_{v\in\Phi_K(\mathfrak{n})\setminus\{v_0\}}\left(\left|m_{v_1j}\right|\prod_{s=2}^{K}\left|m_{v_sv_{s-1}}\right|\right)\left(\prod_{s=1}^{K}\varepsilon_{v_s,\mathfrak{n}-\sum_{r=1}^{s}\mathbf{1}_{v_r},0,t}\right)\Bigg]\\
&\leq \Theta\left[\left(\left|m_{v_{0,1}j}\right|\prod_{s=2}^{K}\left|m_{v_{0,s}v_{0,s-1}}\right|\right)\eta+\sum_{v\in\Phi_K(\mathfrak{n})\setminus\{v_0\}}\left(\left|m_{v_1j}\right|\prod_{s=2}^{K}\left|m_{v_sv_{s-1}}\right|\right)\right]\\
&= \Theta\left[\left(\left|m_{v_{0,1}j}\right|\prod_{s=2}^{K}\left|m_{v_{0,s}v_{0,s-1}}\right|\right)(\eta-1)+\sum_{v\in\Phi_K(\mathfrak{n})}\left(\left|m_{v_1j}\right|\prod_{s=2}^{K}\left|m_{v_sv_{s-1}}\right|\right)\right]\\
&\leq \Theta\left[\delta^K(\eta-1)+\sum_{v\in\Phi_K(\mathfrak{n})}\left(\left|m_{v_1j}\right|\prod_{s=2}^{K}\left|m_{v_sv_{s-1}}\right|\right)\right]\leq\Theta\left[1-\delta^K(1-\eta)\right]=\lambda\Theta. \qquad \blacksquare
\end{aligned}$$

We now obtain the exponential decay of $\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}$ in the set $\mathfrak{N}_c(\sigma)$.

Theorem 3.42. *Suppose that Hypotheses 3.10 and 3.11 are satisfied. Let $T \geq \mu > 0$ and $\sigma \in (0,1)$. Then there exist $C \geq 1$, $\gamma > 0$ and $K \in \mathbb{N}^*$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\sigma)$ with $\min_{i\in[\![1,N]\!]} n_i \geq K$, $i,j \in [\![1,N]\!]$, $x \in [0,L_j]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have*

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq Ce^{-\gamma|\mathfrak{n}|_{\ell^1}}. \tag{3.40}$$

Proof. Take $\rho = \sigma/2$ and let $\lambda \in (0,1)$ and $K \in \mathbb{N}^*$ be as in Lemma 3.41. For $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} n_i \geq K$, we set

$$q_{\max}(\mathfrak{n}) = \max\left\{q \in \mathbb{N} \mid \mathfrak{n}-\mathfrak{r} \in \mathfrak{N}_c(\rho) \text{ for every } \mathfrak{r} \in \mathfrak{N} \text{ with } |\mathfrak{r}|_{\ell^1} = qK\right\}.$$

From Lemma 3.41, one deduces by an immediate inductive argument that for every $q \in [\![1, q_{\max}(\mathfrak{n})]\!]$, $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$, and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we have

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \lambda^q \max_{\substack{\mathfrak{p}\in\mathfrak{N} \mid |\mathfrak{p}|_{\ell^1}=qK\\ p_r\leq n_r,\ \forall r\in[\![1,N]\!]\\ s\in[\![1,N]\!] \mid p_s>0}} \left|\vartheta^{(i)}_{s,\mathfrak{n}-\mathfrak{p},L_s,t}\right|.$$

By Corollary 3.29, it holds $\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq 1$ for every $i,j \in [\![1,N]\!]$, $\mathfrak{n} \in \mathfrak{N}$ and $t \in \mathbb{R}$. Therefore, for every $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i\in[\![1,N]\!]} \mathfrak{n}_i \geq K$, $i,j \in [\![1,N]\!]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T,\mu)$, $k \in [\![1,N_d]\!]$, we obtain

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},L_j,t}\right| \leq \lambda^{q_{\max}(\mathfrak{n})}. \tag{3.41}$$

Notice now that, by definition of $q_{\max}$, one also has that

$$q_{\max}(\mathfrak{n}) + 1 \geq \frac{1}{K}\min\{|\mathfrak{n}-\mathfrak{r}|_{\ell^1} \mid \mathfrak{r} \in \mathfrak{N}_b(\rho)\},$$

where, according to Definition 3.31, $\mathfrak{N}_b(\rho) = \mathfrak{N} \setminus \mathfrak{N}_c(\rho)$. One deduces at once that there exists $\xi > 0$ such that, for every $\mathfrak{n} \in \mathfrak{N}_c(\sigma)$,

$$q_{\max}(\mathfrak{n}) + 1 \geq \xi \left|\mathfrak{n}\right|_{\ell^1}. \tag{3.42}$$

Since $\lambda \in (0,1)$, setting $\gamma = -\xi \log \lambda > 0$ and $C = 1/\lambda$ one concludes by inserting (3.42) into (3.41) and then using (3.20a). ■

3.4.5 Exponential convergence of the solutions

To conclude the proof of Theorem 3.1, it suffices to combine Proposition 3.24 with the estimates of $\vartheta^{(i)}_{j,\mathfrak{n},x,t}$ for $\mathfrak{n} \in \mathfrak{N}_b(\sigma)$ and $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ given by Theorems 3.32 and 3.42, respectively.

Proof of Theorem 3.1. Let $C_1, \gamma_1 > 0$ and $\rho > 0$ be as in Theorem 3.32. Take $\sigma = \rho$ in Theorem 3.42 and let $C_2, \gamma_2 > 0$ and $K \in \mathbb{N}^*$ be as in that theorem. Let

$$\gamma = \min\{\gamma_1, \gamma_2\}, \qquad C = \max\{C_1, C_2, e^{\gamma K/\rho}\}.$$

Take $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0, L_j]$, $t \in \mathbb{R}$ and $\alpha_k \in \mathcal{G}(T, \mu)$, $k \in [\![1, N_d]\!]$. If $\mathfrak{n} \in \mathfrak{N}_b(\rho)$ or $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i \in [\![1,N]\!]} n_i \geq K$, then one concludes directly from Theorems 3.32 and 3.42. Finally, if $\mathfrak{n} \in \mathfrak{N}_c(\rho)$ with $\min_{i \in [\![1,N]\!]} n_i < K$, note that $\left|\mathfrak{n}\right|_{\ell^1} \leq K/\rho$. Then, by Corollary 3.29, one has

$$\left|\vartheta^{(i)}_{j,\mathfrak{n},x,t}\right| \leq 1 \leq C e^{-\gamma K/\rho} \leq C e^{-\gamma \left|\mathfrak{n}\right|_{\ell^1}}.$$

Theorem 3.1 now follows from Proposition 3.24. ■

Remark 3.43. By Remark 3.36, Hypothesis 3.10 can be replaced by Hypothesis 3.37 in Lemma 3.41 and Theorem 3.42, and so the same is also true for Theorem 3.1. We recall that the case $p = +\infty$ also follows from Proposition 3.24 thanks to Remark 3.26.

3.A Well-posedness of the Cauchy problems (3.7), (3.8), and (3.9)

We are interested in this section in the proof of Theorem 3.5, which states the well-posedness of the Cauchy problem (3.7). This is done in two steps. First, we show that the operator A defined in (3.6) is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, thus establishing the well-posedness of the undamped system. We then consider the operator $B(t) = \sum_{i=1}^{N_d} \alpha_i(t) B_i$ as a bounded time-dependent perturbation of A in order to conclude the well-posedness of (3.7).

3.A.1 Preliminaries

Let X be a Banach space and let $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ be an operator in X. The definitions of strong and weak solutions for the Cauchy problem associated with A can be found for instance in [144]. Recall that if A is densely defined with a non-empty resolvent set, then the Cauchy problem associated with A has a unique strong solution for each initial condition $z_0 \in D(A)$ if and only if A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$. In this case, the solution is given by $z(t) = e^{tA} z_0$ (see, for instance, [144, Chapter 4, Theorem 1.3]). Then, $t \mapsto e^{tA} z_0$ is a well-defined continuous function for every $z_0 \in \mathsf{X}$ and it is the unique weak solution of the Cauchy problem associated with A.

Definition 3.44. A family of operators $\{T(t,s)\}_{t\geq s\geq 0} \subset \mathcal{L}(\mathsf{X})$ is an *evolution family* on X if

(a) $T(s,s) = \mathrm{Id}_{\mathsf{X}}$ for every $s \geq 0$,

(b) $T(t,s) = T(t,\tau)T(\tau,s)$ for every $t \geq \tau \geq s \geq 0$,

(c) for every $z \in \mathsf{X}$, $(t,s) \mapsto T(t,s)z$ is continuous for every $t \geq s \geq 0$.

An evolution family is *exponentially bounded* if it satisfies further the following property.

(d) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t,s)\|_{\mathcal{L}(\mathsf{X})} \leq M e^{\omega(t-s)}$ for every $t \geq s \geq 0$.

For references on evolution families see, for instance, [44, 104, 147]. We are interested here in a family of the form $A(t) = A + B(t)$ where A is the generator of a strongly continuous semigroup and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. We shall use here the following notions of solution.

Definition 3.45. Consider the problem

$$\begin{cases} \dot{z}(t) = (A + B(t))z(t), & t \geq s \geq 0, \\ z(s) = z_0, \end{cases} \tag{3.43}$$

where A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$.

(a) We say that $z : [s,+\infty) \to \mathsf{X}$ is a *regular solution* of (3.43) if z is continuous, $z(s) = z_0$, $z(t) \in D(A)$ for every $t \geq s$, z is differentiable for almost every $t \geq s$, $\dot{z} \in L^\infty_{\mathrm{loc}}([s,+\infty), \mathsf{X})$ and $\dot{z}(t) = (A + B(t))z(t)$ for almost every $t \geq s$.

(b) We say that $z : [s,+\infty) \to \mathsf{X}$ is a *mild solution* of (3.43) if z is continuous and, for every $t \geq s$, we have
$$z(t) = e^{(t-s)A} z_0 + \int_s^t e^{(t-\tau)A} B(\tau) z(\tau) d\tau.$$

Here, the integrals of X-valued functions should be understood as Bochner integrals; see, for instance, [175]. In the following proposition, we summarize the main facts needed for the present chapter.

Proposition 3.46. *Let A be the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $B \in L^\infty(\mathbb{R}_+, \mathcal{L}(\mathsf{X}))$. Then, the following holds true:*

(a) *every regular solution of (3.43) is also a mild solution;*

(b) *there exists a unique family $\{T(t,s)\}_{t\geq s\geq 0}$ of bounded operators in X such that $(t,s) \mapsto T(t,s)z$ is continuous for every $z \in \mathsf{X}$ and*
$$T(t,s)z = e^{(t-s)A} z + \int_s^t e^{(t-\tau)A} B(\tau) T(\tau,s) z d\tau, \qquad \forall z \in \mathsf{X}. \tag{3.44}$$

Furthermore, this family is an exponentially bounded evolution family;

(c) *for every $z_0 \in \mathsf{X}$, (3.43) admits a unique mild solution z, given by $z(t) = T(t,s)z_0$.*

3.A.2 Proof of Theorem 3.6

Since $\sum_{i=1}^{N_d} B_i$ is a bounded operator, it suffices to show Theorem 3.6 for A.

Proposition 3.47. *Let $p \in [1,+\infty)$. The operator A is closed and densely defined. Moreover, $D(A)$ endowed with the graph norm is a Banach space compactly embedded in X_p.*

Proof. The proposition follows straightforwardly by the remark that the graph norm on $D(A)$ coincides with the usual norm in $\prod_{i=1}^N W^{1,p}(0,L_i)$, that is,

$$\|z\|_{D(A)}^p = \sum_{i=1}^N \left(\|u_i\|_{L^p(0,L_i)}^p + \left\|u_i'\right\|_{L^p(0,L_i)}^p\right)$$

for $z = (u_1,\dots,u_N) \in D(A)$. ∎

Proposition 3.48. *Let $p \in [1,+\infty)$. The resolvent set $\rho(A)$ of A is nonempty.*

Proof. Since A is closed, we have $\lambda \in \rho(A)$ if and only if $\lambda - A$ is a bijection from $D(A)$ to X_p. A direct computation based on explicit formulas yields that $\lambda - A$ is a bijection as soon as $\lambda \in \mathbb{R}$ with $\lambda > \frac{\log|M|_{\ell^2}}{L_{\min}}$. ∎

We now turn to a result of existence of solutions of (3.8) when $z_0 \in D(A)$.

Theorem 3.49. *For every $z_0 = (u_{1,0},\dots,u_{N,0}) \in D(A)$, there exists a unique strong solution $z = (u_1,\dots,u_N) \in \mathcal{C}^0(\mathbb{R}_+, D(A)) \cap \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}_p)$ of (3.8).*

Proof. Let $T_0 > 0$ be such that $T_0 < L_{\min}$. Note that it suffices to show the theorem for solutions in $\mathcal{C}^0([0,T_0], D(A)) \cap \mathcal{C}^1([0,T_0], \mathsf{X}_p)$, since T_0 does not depend on $z_0 \in D(A)$. Let $z_0 = (u_{1,0},\dots,u_{N,0}) \in D(A)$. It follows easily from the transport equation and the transmission condition (3.2) that a solution $t \mapsto z(t) = (u_1(t),\dots,u_N(t))$ of (3.8) necessarily satisfies

$$u_i(t,x) = \begin{cases} \displaystyle\sum_{j=1}^N m_{ij} u_{j,0}(L_j - t + x) & \text{if } 0 \le x \le t, \\ u_{i,0}(x-t) & \text{if } x > t. \end{cases} \tag{3.45}$$

Conversely, if $z = (u_1,\dots,u_N)$ is given by (3.45), then it solves (3.8) and has z_0 as initial condition. Moreover, one checks by direct computations that z fulfills the required regularity properties. ∎

Proof of Theorem 3.6. From Propositions 3.47 and 3.48 and Theorem 3.49, we obtain that A generates a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$ (see, for instance, [144, Chapter 4, Theorem 1.3]). Since $\sum_{i=1}^{N_d} B_i \in \mathcal{L}(\mathsf{X}_p)$, $A + \sum_{i=1}^{N_d} B_i$ also generates a strongly continuous semigroup (see [144, Chapter 3, Theorem 1.1]). ∎

Remark 3.50. In the particular case $p = 2$ and $|M|_{\ell^2} \le 1$, one may conclude more easily that A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$, without having to construct the explicit formula (3.45) for the solution as we did in Theorem 3.49. Indeed, a straightforward computation shows that, for any $M \in \mathcal{M}_N(\mathbb{R})$, the adjoint operator A^* of A is given by

$$D(A^*) = \left\{ (u_1,\dots,u_N) \in \prod_{i=1}^N H^1(0,L_i) \,\middle|\, u_i(L_i) = \sum_{j=1}^N m_{ji} u_j(0) \right\},$$

$$A^*(u_1,\dots,u_N) = \left(\frac{du_1}{dx},\dots,\frac{du_N}{dx}\right).$$

Also, for any $z = (u_1,\dots,u_N) \in D(A)$, we have

$$\langle z, Az\rangle = -\sum_{i=1}^N \int_0^{L_i} u_i u_i' = \frac{1}{2}\sum_{i=1}^N (u_i(0)^2 - u_i(L_i)^2) \le \frac{|M|_{\ell^2}^2 - 1}{2} \sum_{i=1}^N u_i(L_i)^2,$$

since, by (3.2), we have $\sum_{i=1}^N u_i(0)^2 \le |M|_{\ell^2}^2 \sum_{i=1}^N u_i(L_i)^2$. Thus, if $|M|_{\ell^2} \le 1$, we have $\langle z, Az\rangle \le 0$ for every $z \in D(A)$, so that A is dissipative. A similar computation holds for A^*, with M replaced by its transpose M^{T}, showing that A^* is also dissipative. Hence A generates a strongly continuous semigroup of contractions $\{e^{tA}\}_{t\ge 0}$ (see, for instance, [144, Chapter 1, Theorem 4.4]).

3.A.3 Proof of Theorem 3.5

Thanks to Theorem 3.6 and Proposition 3.46(b) and (c), there exists a unique evolution family $\{T(t,s)\}_{t\ge s\ge 0}$ associated with (3.7) such that, for every $z_0 \in \mathsf{X}_p$, $t \mapsto T(t,s)z_0$ is the unique mild solution of (3.7) with initial condition $z(s) = z_0$. In order to complete the proof of Theorem 3.5, it suffices to show that this solution is actually regular when $z_0 \in D(A)$. To do so, we study the explicit formula for the solutions of (3.7) for small time, as we did with the undamped system (3.8) in Theorem 3.49.

Proof of Theorem 3.5. Let $T_0 > 0$ be such that $T_0 < L_{\min}$. As in Theorem 3.49, it suffices to show the theorem for solutions in $\mathcal{C}^0([s, s+T_0], \mathsf{X}_p)$. Since the class $L^\infty(\mathbb{R}, [0,1])$ is invariant by time-translation, we can also suppose without loss of generality that $s = 0$. In order to simplify the notations, we define the function $\varphi_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+^*$ for $i \in [\![1, N]\!]$ by

$$\varphi_i(t,x) = e^{-\int_0^t \alpha_i(s)\chi_i(x-t+s)ds},$$

where we extend the function χ_i to $\mathbb{R}$ by 0 outside its interval of definition $[0, L_i]$. (In particular, $\varphi_i \equiv 1$ for $i \in [\![N_d+1, N]\!]$.) We have that both φ_i and $1/\varphi_i$ belong to $L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap \mathcal{C}^0(\mathbb{R}_+ \times \mathbb{R})$ and to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R})$.

Let $z_0 = (u_{1,0}, \dots, u_{N,0}) \in D(A)$. We claim that, for $0 \le t \le T_0$, the function $t \mapsto z(t) = (u_1(t), \dots, u_N(t))$ given by

$$u_i(t,x) = \begin{cases} \dfrac{\varphi_i(t,x)}{\varphi_i(t-x,0)} \displaystyle\sum_{j=1}^N m_{ij}\varphi_j(t-x, L_j)u_{j,0}(x+L_j-t), & \text{if } 0 \le x \le t, \\ \varphi_i(t,x)u_{i,0}(x-t), & \text{if } x > t, \end{cases}$$

is in $\mathcal{C}^0([0,T_0], \mathsf{X}_p)$, $z(0) = z_0$, $z(t) \in D(A)$ for every $t \in [0, T_0]$, z is differentiable for almost every $t \in [0,T_0]$, $\dot z \in L^\infty([0,T_0], \mathsf{X}_p)$ and $\dot z(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t)B_i z(t)$ for almost every $t \in [0, T_0]$. Indeed, it is clear that z is well-defined, $z(0) = z_0$, and $z(t) \in \mathsf{X}_p$ for every $t \in [0,T_0]$. It is also clear, thanks to the regularity properties of φ_i, that, for every $t \in [0, T_0]$, $u_i(t) \in W^{1,p}(0,t)$ and $u_i(t) \in W^{1,p}(t, T_0)$, and, since $x \mapsto u_i(t,x)$ is continuous at $x = t$, we conclude that $u_i(t) \in W^{1,p}(0, L_i)$. Furthermore,

$$u_i(t,0) = \sum_{j=1}^N m_{ij}\varphi_j(t, L_j)u_{j,0}(L_j - t) = \sum_{j=1}^N m_{ij}u_j(t, L_j),$$

and thus $z(t) \in D(A)$ for every $t \in [0, T_0]$. By the same argument, we also obtain that $u_i(\cdot, x) \in W^{1,p}(0, T_0)$ for every $x \in [0, L_i]$. Computing $u_i(t+h, x) - u_i(t,x)$ for $t, t+h \in [0, T_0]$ also shows, by a straightforward estimate, that $\|u_i(t+h) - u_i(t)\|_{L^p(0,L_i)} \to 0$ as $h \to 0$, and thus $z \in \mathcal{C}^0([0,T_0], \mathsf{X}_p)$.

Since $u_i(\cdot, x) \in W^{1,p}(0,T_0)$ for every $x \in [0, L_i]$, one can also compute $\partial_t u_i(t,x)$, and it is easy to verify that $\partial_t u_i \in L^\infty([0,T_0], L^p(0, L_i))$. Hence z is differentiable almost everywhere, with $\dot z = (\partial_t u_1, \dots, \partial_t u_N) \in L^\infty([0,T_0], \mathsf{X}_p)$.

Notice now that $\dot{z}(t) - Az(t) = (\partial_t u_1(t) + \partial_x u_1(t), \dots, \partial_t u_N(t) + \partial_x u_N(t))$ is given by

$$\partial_t u_i(t,x) + \partial_x u_i(t,x) = \begin{cases} \dfrac{\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x)}{\varphi_i(t-x,0)} \displaystyle\sum_{j=1}^{N} m_{ij} \varphi_j(t-x, L_j) u_{j,0}(x + L_j - t), & \text{if } 0 \leq x \leq t, \\ [\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x)] u_{i,0}(x-t), & \text{if } x > t, \end{cases}$$

and one can compute that $\partial_t \varphi_i(t,x) + \partial_x \varphi_i(t,x) = -\alpha_i(t)\chi_i(x)\varphi_i(t,x)$ almost everywhere, so that

$$\partial_t u_i(t,x) + \partial_x u_i(t,x) = -\alpha_i(t)\chi_i(x) u_i(t,x).$$

Thus $\dot{z}(t) - Az(t) = -\sum_{i=1}^{N_d} \alpha_i(t) B_i z(t)$ for almost every $t \in [0, T_0]$, which concludes the proof of existence. Uniqueness results from the fact that every regular solution is in particular a mild solution, which is unique, according to Proposition 3.46. ■

3.B Asymptotic behavior of (3.11)

We consider here System (3.11) from Example 3.8. Let X_2 be the Hilbert space $\mathsf{X}_2 = L^2(0, L_1) \times L^2(0, L_2)$. The goal of this section is to prove Theorem 3.9 concerning the asymptotic behavior of (3.11) when $\chi \equiv 0$, and also to show the existence of periodic solutions to (3.11) with a persistent damping when $L_1/L_2 \in \mathbb{Q}$ and $b - a$ is small enough. The proof of Theorem 3.9(a) being based on LaSalle Principle, we recall its formulation in a Banach space in Section 3.B.1.

3.B.1 LaSalle Principle in a Banach space

In this section, X denotes a Banach space and $A : D(A) \subset \mathsf{X} \to \mathsf{X}$ is a linear operator in X that generates a strongly continuous semigroup $\{e^{tA}\}_{t \geq 0}$.

Definition 3.51.

(a) For $z_0 \in \mathsf{X}$, the *ω-limit set* $\omega(z_0)$ is the set of $z \in \mathsf{X}$ such that there exists a nondecreasing sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_n A} z_0 \to z$ in X as $n \to \infty$. A set $M \subset \mathsf{X}$ is *positively invariant* under $\{e^{tA}\}_{t \geq 0}$ if, for every $z_0 \in M$ and $t \geq 0$, we have $e^{tA} z_0 \in M$. For $E \subset \mathsf{X}$, the *maximal positively invariant subset* M of E is the union of all positively invariant sets contained in E.

(b) A *Lyapunov function* for $\{e^{tA}\}_{t \geq 0}$ is a continuous function $V : \mathsf{X} \to \mathbb{R}_+$ such that

$$\dot{V}(z) = \limsup_{t \to 0+} \frac{V(e^{tA} z) - V(z)}{t} \leq 0, \qquad \forall z \in \mathsf{X}.$$

The following results can be found in [81, 95, 161].

Theorem 3.52.

(a) *Suppose that $\{e^{tA} z_0 \mid t \geq 0\}$ is precompact in X. Then $\omega(z_0)$ is a nonempty, compact, connected, positively invariant set.*

(b) *Let V be a Lyapunov function on X, define $E = \{z \in \mathsf{X} \mid \dot{V}(z) = 0\}$ and let M be the maximal positively invariant subset of E. If $\{e^{tA} z_0 \mid t \geq 0\}$ is precompact in X, then $\omega(z_0) \subset M$.*

3.B.2 Asymptotic behavior when $L_1/L_2 \notin \mathbb{Q}$

Let us turn to the proof of Theorem 3.9(a). We consider the undamped system

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \in \mathbb{R}_+,\ x \in [0, L_i],\ i \in \{1,2\}, \\ u_1(t,0) = u_2(t,0) = \dfrac{u_1(t,L_1) + u_2(t,L_2)}{2}, & t \in \mathbb{R}_+, \\ u_i(0,x) = u_{i,0}(x), & x \in [0,L_i],\ i \in \{1,2\}. \end{cases} \tag{3.46}$$

Let $V : \mathsf{X}_2 \to \mathbb{R}$ be the function $V(z) = \|z\|_{\mathsf{X}_2}^2$ and A be the operator given in Definition 3.3 in the case $p = 2$, $N = 2$ and $m_{ij} = 1/2$ for $i, j \in \{1,2\}$, which is the operator associated with System (3.46). By Theorem 3.6, A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$.

Lemma 3.53. *The function V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. If $z = (u_1, u_2) \in D(A)$, we have*

$$\dot{V}(z) = 2\langle z, Az\rangle = -\frac{(u_1(L_1) - u_2(L_2))^2}{2}.$$

Proof. Notice first that, for $z = (u_1,u_2) \in D(A)$, we have $\langle z, Az\rangle = -(u_1(L_1) - u_2(L_2))^2/4 \leq 0$. Take $z \in D(A)$, so that $t \mapsto e^{tA}z$ is continuously differentiable in $\mathbb{R}_+$; thus $t \mapsto V(e^{tA}z)$ is continuously differentiable in $\mathbb{R}_+$ with $\frac{d}{dt}V(e^{tA}z) = 2\left\langle e^{tA}z, Ae^{tA}z\right\rangle$. Hence, for every $z \in D(A)$, $\dot{V}(z) = 2\langle z, Az\rangle \leq 0$. This also shows that $\left\|e^{tA}z\right\|_{\mathsf{X}_2} \leq \|z\|_{\mathsf{X}_2}$ for every $z \in D(A)$ and $t \geq 0$, and, by the density of $D(A)$ in X_2, we obtain that $\left\|e^{tA}z\right\|_{\mathsf{X}_2} \leq \|z\|_{\mathsf{X}_2}$ for every $z \in \mathsf{X}_2$ and $t \geq 0$, i.e., $\{e^{tA}\}_{t\geq 0}$ is a contraction semigroup. Thus $\dot{V}(z) \leq 0$ for every $z \in \mathsf{X}_2$, and so V is a Lyapunov function for $\{e^{tA}\}_{t\geq 0}$. ∎

It is then immediate to prove the following.

Lemma 3.54. *For every $z \in D(A)$, $\{e^{tA}z \mid t \geq 0\}$ is precompact in X_2 and $\omega(z) \subset D(A)$.*

Set $E = \{z \in \mathsf{X}_2 \mid \dot{V}(z) = 0\}$ and let M be the maximal positively invariant subset of E.

Lemma 3.55. *Suppose $L_1/L_2 \notin \mathbb{Q}$. Then*

$$D(A) \cap M = \{(\lambda, \lambda) \in L^2(0,L_1) \times L^2(0,L_2) \mid \lambda \in \mathbb{R}\},$$

i.e., $D(A) \cap M$ is the set of constant functions on $L^2(0,L_1) \times L^2(0,L_2)$.

Proof. Take $z_0 = (u_{1,0}, u_{2,0}) \in D(A) \cap M$. By the positive invariance of M, $e^{tA}z_0 \in D(A) \cap M$ for every $t \geq 0$.

Let us denote $z(t) = (u_1(t), u_2(t)) = e^{tA}z_0$, which is a strong solution of (3.46) with initial condition z_0. Since $z(t) \in M$, we have $\dot{V}(z(t)) = 0$ for every $t \geq 0$, which means, by Lemma 3.53, that $u_1(t,L_1) = u_2(t,L_2)$ for every $t \geq 0$. Then we have that

$$u_1(t,0) = u_2(t,0) = \frac{u_1(t,L_1) + u_2(t,L_2)}{2} = u_1(t,L_1) = u_2(t,L_2), \qquad \forall t \in \mathbb{R}_+.$$

Without any loss of generality, we can suppose that $L_1 \leq L_2$. For $t \geq L_1$ and $x \in [0, L_1]$, we have that

$$u_1(t,x) = u_1(t-x,0) = u_2(t-x,0) = u_2(t,x),$$

and so $u_1(t,x) = u_2(t,x)$. Now, for $t \geq L_1$ and $x \in [0,L_1]$, we have that

$$u_1(t+L_1,x) = u_1(t+L_1-x,0) = u_1(t+L_1-x,L_1) = u_1(t-x,0) = u_1(t,x)$$

and thus $[L_1,+\infty) \ni t \mapsto u_1(t,x)$ is a L_1-periodic function for every $x \in [0,L_1]$. Similarly, for $t \geq L_2$ and $x \in [0,L_2]$, we have that

$$u_2(t+L_2,x) = u_2(t+L_2-x,0) = u_2(t+L_2-x,L_2) = u_2(t-x,0) = u_2(t,x)$$

and thus $[L_2,+\infty) \ni t \mapsto u_2(t,x)$ is a L_2-periodic function for every $x \in [0,L_2]$. Since $u_1(t,x) = u_2(t,x)$ for $t \geq L_1$, $x \in [0,L_1]$, we obtain that $[L_2,+\infty) \ni t \mapsto u_1(t,x)$ is both L_1-periodic and L_2-periodic for every $x \in [0,L_1]$, and the fact that $L_1/L_2 \notin \mathbb{Q}$ thus implies that $[L_2,+\infty) \ni t \mapsto u_1(t,x) = u_2(t,x) =: \lambda(x)$ is constant for every $x \in [0,L_1]$. Clearly, $\lambda(x)$ does not depend on x since

$$\lambda(x) = u_1(t,x) = u_1(t-x,0) = \lambda(0), \qquad \forall t \geq L_2+L_1,\ \forall x \in [0,L_1],$$

and so we shall denote this constant value simply by λ. We thus have that

$$u_1(t,x) = u_2(t,x) = \lambda, \qquad \forall t \geq L_2,\ \forall x \in [0,L_1].$$

We deduce at once that $u_{1,0}$ and $u_{2,0}$ are both equal to the constant function λ and hence

$$D(A) \cap M \subset \{(\lambda,\lambda) \in L^2(0,L_1) \times L^2(0,L_2) \mid \lambda \in \mathbb{R}\}.$$

The converse inclusion is trivial and this concludes the proof. ∎

Suppose now that $z_0 \in D(A)$. By Lemma 3.54 and Theorem 3.52(b), we have that $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma 3.55, if $L_1/L_2 \notin \mathbb{Q}$, we get that every function in $\omega(z_0)$ is constant. We now wish to show that $\omega(z_0)$ contains only one point in X_2, which will imply that $e^{tA}z_0$ converges to this function as $t \to +\infty$. To do so, we study a conservation law for (3.46).

We define $U : \mathsf{X}_2 \to \mathbb{R}$ by

$$U(u_1,u_2) = \frac{1}{L_1+L_2}\left(\int_0^{L_1} u_1(x)dx + \int_0^{L_2} u_2(x)dx\right).$$

Notice that U is well defined and continuous in X_2 since we have the continuous embedding $\mathsf{X}_2 \hookrightarrow L^1(0,L_1) \times L^1(0,L_2)$.

Lemma 3.56. *For every $z \in \mathsf{X}_2$ and $t \geq 0$, we have $U(e^{tA}z) = U(z)$.*

Proof. By the density of $D(A)$ in X_2 and by the continuity of U, it suffices to show this for $z \in D(A)$. In this case, the function $t \mapsto U(e^{tA}z)$ is differentiable in $\mathbb{R}_+$, and, noting $e^{tA}z = (u_1(t),u_2(t))$, we have by a trivial computation that $\frac{d}{dt}U(e^{tA}z) = 0$. ∎

Define the operator L on X_2 by $Lz = (U(z),U(z))$ and notice that $L \in \mathcal{L}(\mathsf{X}_2)$. The main result of this section is the following, which proves Theorem 3.9(a) and gives the explicit value of the constant λ.

Theorem 3.57. *Suppose $L_1/L_2 \notin \mathbb{Q}$. Then, for every $z_0 \in \mathsf{X}_2$, $\lim_{t\to+\infty} e^{tA}z_0 = Lz_0$.*

Proof. Since L is a bounded operator and the semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly bounded, it suffices by density to show this result for $z_0 \in D(A)$. By Lemma 3.54 and Theorem 3.52(b), we have $\omega(z_0) \subset D(A) \cap M$ and thus, by Lemma 3.55, every function in $\omega(z_0)$ is constant. Let $z = (\lambda,\lambda) \in \omega(z_0)$ with $\lambda \in \mathbb{R}$ and take $(t_n)_{n\in\mathbb{N}}$ a nondecreasing sequence in $\mathbb{R}_+$ with $t_n \to +\infty$ as $n \to \infty$ such that $e^{t_nA}z_0 \to z$ in X_2 as $n \to \infty$. By the continuity of U and by Lemma 3.56, we obtain that

$$\lambda = U(z) = \lim_{n\to\infty} U(e^{t_nA}z_0) = U(z_0)$$

and thus $z = Lz_0$. Hence $\omega(z_0) = \{Lz_0\}$ and, by definition of $\omega(z_0)$, this shows that $e^{tA}z_0 \to Lz_0$ as $t \to +\infty$, which gives the desired result. ∎

3.B.3 Periodic solutions for the undamped system

We now turn to a constructive proof of Theorem 3.9(b).

Proof of Theorem 3.9(b). Take $p, q \in \mathbb{N}^*$ coprime such that $L_1/L_2 = p/q$. Let $\ell = L_1/p = L_2/q$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with support included in $(0, \ell)$. For $x \in [0, L_1]$, we define $u_{1,0}$ by

$$u_{1,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - k\ell). \tag{3.47}$$

Notice that, for each $x \in \mathbb{R}$, there exists at most one $k \in \mathbb{Z}$ such that $\varphi(x-k\ell) \neq 0$. In particular, $u_{1,0} \in \mathcal{C}^\infty([0, L_1])$. Similarly we define $u_{2,0} \in \mathcal{C}^\infty([0, L_2])$ by the same expression

$$u_{2,0}(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - k\ell), \quad x \in [0, L_2]. \tag{3.48}$$

Define $u_j(t, x) = u_{j,0}(x - t)$ for $j = 1, 2$. Since $L_1 = p\ell$, $L_2 = q\ell$, we have

$$u_1(t, L_1) = u_2(t, L_2) = u_1(t, 0) = u_2(t, 0).$$

Thus, (u_1, u_2) is the unique solution of (3.46) with initial data $z_0 = (u_{1,0}, u_{2,0})$. It is periodic in time, and non-constant if φ is chosen to be non-constant. ■

3.B.4 Periodic solutions for the persistently damped system

Sections 3.B.2 and 3.B.3 present the asymptotic behavior of (3.46) in the cases $L_1/L_2 \notin \mathbb{Q}$ and $L_1/L_2 \in \mathbb{Q}$, showing that all solutions converge to a constant in the first case and that periodic solutions exist in the second one. When considering System (3.11) with a persistent damping, the fact that all its solutions converge exponentially to the origin when $L_1/L_2 \notin \mathbb{Q}$ is a consequence of our main result, Theorem 3.1. However, if $L_1/L_2 \in \mathbb{Q}$ and the damping interval $[a, b]$ is small enough, one may obtain periodic solutions, thus showing that Theorem 3.1 cannot hold in general for $L_1/L_2 \in \mathbb{Q}$ and any length of damping interval.

Theorem 3.58. *Suppose that $L_1/L_2 \in \mathbb{Q}$. Then there exists $\ell_0 > 0$ such that, if $b - a \leq \ell_0$, there exists $\alpha \in \mathcal{G}(4\ell_0, \ell_0)$ for which (3.11) admits a non-zero periodic solution.*

Proof. We consider here the construction of a periodic solution for (3.46) done in the proof of Theorem 3.9(b). Take $p, q \in \mathbb{N}^*$ coprime such that $L_1/L_2 = p/q$ and note $\ell = L_1/p = L_2/q$. Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ not identically zero with support included in $(0, \ell/2)$. By proceeding as in Theorem 3.9(b) we get a periodic non-zero solution (u_1, u_2) of (3.46) given by

$$u_1(t, x) = \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell), \qquad u_2(t, x) = \sum_{k=-\infty}^{+\infty} \varphi(x - t - k\ell). \tag{3.49}$$

Take $\ell_0 = \ell/4$ and suppose that $b - a \leq \ell_0$. We construct a periodic signal $\alpha : \mathbb{R} \to \{0, 1\}$ defined by

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{n \in \mathbb{Z}} [a - (n + 1/2)\ell, b - n\ell], \\ 1, & \text{otherwise.} \end{cases}$$

This defines a periodic signal α with period $T = \ell = 4\ell_0$ which belongs to $\mathcal{G}(4\ell_0, \ell_0)$. One then easily checks that $\alpha(t)\chi(x)u_2(t, x) = 0$ for every $(t, x) \in \mathbb{R}_+ \times [0, L_2]$, and so (3.49) satisfies (3.11). ■

3.C Proof of Theorem 3.15

From now on, we use the convention $\beta^{(i)}_{j,\mathfrak{n}} = 0$ if $\mathfrak{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$ is such that $n_k < 0$ for some index $k \in [\![1, N]\!]$, so that (3.16b) can be written as

$$\beta^{(i)}_{j,\mathfrak{n}} = \sum_{k=1}^{N} m_{kj} \beta^{(i)}_{k,\mathfrak{n}-\mathbf{1}_k}.$$

One then gets by a trivial induction the following result.

Lemma 3.59. *For every $i, j \in [\![1, N]\!]$ and $\mathfrak{n} = (n_1, \dots, n_N) \in \mathfrak{N} \setminus \{\mathbf{0}\}$, we have*

$$\beta^{(i)}_{j,\mathfrak{n}} = \sum_{k=1}^{N} m_{ik} \beta^{(k)}_{j,\mathfrak{n}-\mathbf{1}_k}. \tag{3.50}$$

We can now turn to the proof of Theorem 3.15.

Proof of Theorem 3.15. Let $z_0 = (u_{1,0}, \dots, u_{N,0}) \in D(A)$ and let $z = (u_1, \dots, u_N)$ be defined by (3.14), with $u_i(t, 0)$ given by (3.15). Notice that $u_i(\cdot, 0)$ is defined everywhere on $\mathbb{R}_+$ and is measurable, so that u_i is defined everywhere on $\mathbb{R}_+ \times [0, L_i]$ and is measurable. Note also that u_i is well defined, since

$$u_i(0,0) = \sum_{j=1}^{N} \beta^{(i)}_{j,\mathbf{0}} u_{j,0}(L_j) = u_{i,0}(0)$$

thanks to the fact that $\beta^{(i)}_{j,\mathbf{0}} = m_{ij}$ and $z_0 \in D(A)$.

Let $T_0 > 0$ be as in the proof of Theorem 3.49. The unique solution of (3.3) with initial condition z_0 is then given by (3.45) for $0 \leq t \leq T_0$, and, in order to prove the theorem, it suffices to show that, for every t, τ with $0 \leq \tau \leq t \leq \tau + T_0$, we have

$$u_i(t,x) = \begin{cases} \displaystyle\sum_{j=1}^{N} m_{ij} u_j(\tau, L_j - (t-\tau) + x), & \text{if } 0 \leq x \leq t - \tau, \\ u_i(\tau, x - t + \tau), & \text{if } x > t - \tau. \end{cases} \tag{3.51}$$

Indeed, if this is proved, we apply it to $\tau = 0$ to obtain that z is the solution of (3.3) with initial condition z_0 for $0 \leq t \leq T_0$, and also that $z(t) \in D(A)$ for every $t \in [0, T_0]$, and a simple induction allows us to conclude.

We prove (3.51) by considering three cases.

Case 1. $0 \leq t - \tau < x$ and $t \leq x$. By (3.14), we have $u_i(t,x) = u_{i,0}(x-t) = u_i(\tau, x - t + \tau)$.

Case 2. $0 \leq t - \tau < x$ and $t > x$. Since $x - t + \tau < \tau$, we have in this case $u_i(\tau, x - t + \tau) = u_i(\tau - x + t - \tau, 0) = u_i(t - x, 0) = u_i(t, x)$.

Case 3. $t - \tau \geq x$. We notice first that it suffices to consider the case $x = 0$. Indeed, if $t - \tau \geq x$, then $t \geq x$, and it follows clearly by the definition (3.14) of u_i that $u_i(t,x) = u_i(t-x, 0)$ for $t \geq x \geq 0$. On the other hand, let us denote

$$v_{i,\tau}(t,x) = \begin{cases} \displaystyle\sum_{j=1}^{N} m_{ij} u_j(\tau, L_j - (t-\tau) + x) & \text{if } 0 \leq x \leq t - \tau, \\ u_i(\tau, x - t + \tau) & \text{if } x > t - \tau. \end{cases}$$

It is also clear that $v_{i,\tau}(t,x) = v_{i,\tau}(t-x,0)$, and thus it suffices to show that $u_i(t,0) = v_{i,\tau}(t,0)$ for every $t \in [\tau, \tau+T_0]$ in order to conclude that $u_i(t,x) = u_i(t-x,0) = v_{i,\tau}(t-x,0) = v_{i,\tau}(t,x)$ for every $t \in [\tau, \tau+T_0]$ and $x \in [0, L_i]$ with $t - \tau \geq x$.

For $t \in [\tau, \tau+T_0]$, we have $v_{i,\tau}(t,0) = \sum_{j=1}^N m_{ij} u_j(\tau, L_j - t + \tau)$. Furthermore

$$u_j(\tau, L_j - t + \tau) = \begin{cases} u_{j,0}(L_j - t), & \text{if } t \leq L_j, \\ u_j(t - L_j, 0), & \text{if } t \geq L_j, \end{cases}$$

and so

$$\begin{aligned}
v_{i,\tau}(t,0) &= \sum_{j=1}^N m_{ij} u_j(\tau, L_j - t + \tau) = \sum_{\substack{j=1 \\ L_j \leq t}}^N m_{ij} u_j(t - L_j, 0) + \sum_{\substack{j=1 \\ L_j > t}}^N m_{ij} u_{j,0}(L_j - t) \\
&= \sum_{\substack{j=1 \\ L_j \leq t}}^N m_{ij} \sum_{k=1}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \leq t - L_j}} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) + \sum_{\substack{j=1 \\ L_j > t}}^N m_{ij} u_{j,0}(L_j - t) \\
&= \sum_{\substack{k=1 \\ L_k \leq t}}^N \sum_{\substack{j=1 \\ L_j \leq t}}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \leq t - L_j}} m_{ij} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) \\
&\quad + \sum_{\substack{k=1 \\ L_k > t}}^N \sum_{\substack{j=1 \\ L_j \leq t}}^N \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_k \\ L(\mathfrak{n}) \leq t - L_j}} m_{ij} \beta^{(j)}_{k, \mathfrak{n}} u_{k,0}(L_k - (t - L_j - L(\mathfrak{n}))) + \sum_{\substack{j=1 \\ L_j > t}}^N \beta^{(i)}_{j, \mathbf{0}} u_{j,0}(L_j - t).
\end{aligned}$$

Set

$$\begin{aligned}
A_1(t) &= \{(k,j,\mathfrak{n}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k \leq t,\ L_j \leq t,\ \mathfrak{n} \in \mathfrak{N}_k,\ L(\mathfrak{n}) \leq t - L_j\}, \\
A_2(t) &= \{(k,j,\mathfrak{n}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k > t,\ L_j \leq t,\ \mathfrak{n} \in \mathfrak{N}_k,\ L(\mathfrak{n}) \leq t - L_j\},
\end{aligned}$$

so that we can write

$$\begin{aligned}
v_{i,\tau}(t,0) = &\sum_{(k,j,\mathfrak{n}) \in A_1(t)} m_{ij} \beta^{(j)}_{k, \mathfrak{n} + \left\lfloor \frac{t - L_j - L(\mathfrak{n})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L_j - L(\mathfrak{n})\}_{L_k}) \\
&+ \sum_{(k,j,\mathfrak{n}) \in A_2(t)} m_{ij} \beta^{(j)}_{k, \mathfrak{n}} u_{k,0}(L_k - (t - L_j - L(\mathfrak{n}))) + \sum_{\substack{j=1 \\ L_j > t}}^N \beta^{(i)}_{j, \mathbf{0}} u_{j,0}(L_j - t).
\end{aligned} \tag{3.52}$$

Set

$$\begin{aligned}
B_1(t) = \{(k,j,\mathfrak{m}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k \leq t,\ \mathfrak{m} \in \mathfrak{N}_k,& \\
L(\mathfrak{m}) \leq t,\ \mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k = (r_1, \dots, r_N) \text{ with } r_j \geq 1 &\Big\},
\end{aligned}$$

$$B_2(t) = \{(k,j,\mathfrak{m}) \in [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \mid L_k > t,\ \mathfrak{m} = (m_1, \dots, m_N) \in \mathfrak{N}_k \setminus \{\mathbf{0}\},\ m_j \geq 1,\ L(\mathfrak{m}) \leq t\},$$

and define the functions $\varphi_\lambda : [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N} \to [\![1,N]\!] \times [\![1,N]\!] \times \mathfrak{N}$, $\lambda = 1, 2$, by

$$\varphi_1(k,j,\mathfrak{n}) = \begin{cases} (k,j,\mathfrak{n} + \mathbf{1}_j) & \text{if } k \neq j, \\ (k,j,\mathfrak{n}) & \text{if } k = j, \end{cases} \qquad \varphi_2(k,j,\mathfrak{n}) = (k,j,\mathfrak{n} + \mathbf{1}_j).$$

We claim that φ_λ is a bijection from $A_\lambda(t)$ to $B_\lambda(t)$, $\lambda = 1, 2$. Indeed, it is easy to verify that the image of $A_\lambda(t)$ by φ_λ is included in $B_\lambda(t)$ and that $\varphi_\lambda : A_\lambda(t) \to B_\lambda(t)$ is injective for $\lambda = 1, 2$. Let us check that these functions are surjective.

If $(k, j, \mathfrak{m}) \in B_1(t)$, we note $(r_1, \dots, r_N) = \mathfrak{m} + \lfloor (t - L(\mathfrak{m}))/L_k \rfloor \mathbf{1}_k$ and we set $\mathfrak{n} = \mathfrak{m}$ if $k = j$ and $\mathfrak{n} = \mathfrak{m} - \mathbf{1}_j$ if $k \neq j$. Notice first that, if $k \neq j$, then $m_j = r_j \geq 1$, so that $\mathfrak{m} - \mathbf{1}_j \in \mathfrak{N}$, and thus, in both cases $k = j$ and $k \neq j$, we have $(k, j, \mathfrak{n}) \in [\![1, N]\!] \times [\![1, N]\!] \times \mathfrak{N}$, and clearly $\varphi_1(k, j, \mathfrak{n}) = (k, j, \mathfrak{m})$, so that, in order to conclude that $\varphi_1 : A_1(t) \to B_1(t)$ is surjective, it suffices to show that $(k, j, \mathfrak{n}) \in A_1(t)$. We clearly have $L_k \leq t$ and $\mathfrak{n} \in \mathfrak{N}_k$. If $j = k$, we have $L_j = L_k \leq t$ and, since $r_j = r_k = \lfloor (t - L(\mathfrak{m}))/L_k \rfloor$ and $r_j \geq 1$, we have $(t - L(\mathfrak{m}))/L_k \geq 1$, i.e., $L(\mathfrak{n}) = L(\mathfrak{m}) \leq t - L_k = t - L_j$, so that $(k, j, \mathfrak{n}) \in A_1(t)$. If $j \neq k$, we have $m_j = r_j \geq 1$, so that $L_j \leq m_j L_j \leq L(\mathfrak{m}) \leq t$; also, $L(\mathfrak{n}) = L(\mathfrak{m}) - L_j \leq t - L_j$, so that $(k, j, \mathfrak{n}) \in A_1(t)$. Hence $(k, j, \mathfrak{n}) \in A_1(t)$ in both cases, and thus $\varphi_1 : A_1(t) \to B_1(t)$ is surjective.

If $(k, j, \mathfrak{m}) \in B_2(t)$, we set $\mathfrak{n} = \mathfrak{m} - \mathbf{1}_j$, so that $\mathfrak{n} \in \mathfrak{N}$ and $\varphi_2(k, j, \mathfrak{n}) = (k, j, \mathfrak{m})$. Now, it is clear that $L_k > t$ and $\mathfrak{n} \in \mathfrak{N}_k$, and we have $L_j \leq m_j L_j \leq L(\mathfrak{m}) \leq t$ and $L(\mathfrak{n}) = L(\mathfrak{m}) - L_j \leq t - L_j$. Hence $(k, j, \mathfrak{n}) \in A_2(t)$, and thus $\varphi_2 : A_2(t) \to B_2(t)$ is surjective.

Thanks to the bijections $\varphi_\lambda : A_\lambda(t) \to B_\lambda(t)$, $\lambda = 1, 2$, we can rewrite (3.52) as

$$
\begin{aligned}
v_{i,\tau}(t, 0) = & \sum_{(k,j,\mathfrak{m}) \in B_1(t)} m_{ij} \beta^{(j)}_{k, \mathfrak{m} - \mathbf{1}_j + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\
& + \sum_{(k,j,\mathfrak{m}) \in B_2(t)} m_{ij} \beta^{(j)}_{k, \mathfrak{m} - \mathbf{1}_j} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{j=1 \\ L_j > t}}^{N} \beta^{(i)}_{j, \mathbf{0}} u_{j,0}(L_j - t),
\end{aligned}
$$

and so, by applying Lemma 3.59, we obtain

$$
\begin{aligned}
v_{i,\tau}(t, 0) = & \sum_{\substack{k=1 \\ L_k \leq t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \sum_{\substack{j=1 \\ \left(\mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k\right)_j \geq 1}}^{N} m_{ij} \beta^{(j)}_{k, \mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k - \mathbf{1}_j} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\
& + \sum_{\substack{k=1 \\ L_k > t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \setminus \{\mathbf{0}\} \\ L(\mathfrak{m}) \leq t}} \sum_{\substack{j=1 \\ m_j \geq 1}}^{N} m_{ij} \beta^{(j)}_{k, \mathfrak{m} - \mathbf{1}_j} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{k=1 \\ L_k > t}}^{N} \beta^{(i)}_{k, \mathbf{0}} u_{k,0}(L_k - t) \\
= & \sum_{\substack{k=1 \\ L_k \leq t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k, \mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) \\
& + \sum_{\substack{k=1 \\ L_k > t}}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \setminus \{\mathbf{0}\} \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k, \mathfrak{m}} u_{k,0}(L_k - (t - L(\mathfrak{m}))) + \sum_{\substack{k=1 \\ L_k > t}}^{N} \beta^{(i)}_{k, \mathbf{0}} u_{k,0}(L_k - t) \\
= & \sum_{k=1}^{N} \sum_{\substack{\mathfrak{m} \in \mathfrak{N}_k \\ L(\mathfrak{m}) \leq t}} \beta^{(i)}_{k, \mathfrak{m} + \left\lfloor \frac{t - L(\mathfrak{m})}{L_k} \right\rfloor \mathbf{1}_k} u_{k,0}(L_k - \{t - L(\mathfrak{m})\}_{L_k}) = u_i(t, 0).
\end{aligned}
$$

Hence $v_{i,\tau}(t, 0) = u_i(t, 0)$ for every $t \in [\tau, \tau + T_0]$, which thus concludes the proof of (3.51). ■

3.D A combinatorial estimate

In order to estimate the right-hand side of (3.30), one needs the following lemma.

Lemma 3.60. *Let $\nu \in (0,1)$. There exist $\rho \in (0, 1/2)$, $C, \gamma > 0$ such that, for every $n \in \mathbb{N}$ and $k \in [\![0, \rho n]\!]$, we have*

$$\binom{n}{k}\nu^n \leq Ce^{-\gamma n}. \tag{3.53}$$

Proof. For $n \in \mathbb{N}$, consider the function $f_n(k) = \binom{n}{k}\nu^n$ defined for $k \in [\![0, n]\!]$. Since $k \mapsto \binom{n}{k}$ is increasing for $k \in [\![0, n/2]\!]$, if $k \in [\![0, \rho n]\!]$ for a certain $\rho \in (0, 1/2)$, then

$$f_n(k) \leq f_n(\lfloor \rho n \rfloor). \tag{3.54}$$

Let us estimate $f_n(\lfloor \rho n \rfloor)$ for n large. As $n \to +\infty$, using Stirling's approximation $\log n! = n \log n - n + O(\log n)$, we get

$$\begin{aligned}
\log f_n(\lfloor \rho n \rfloor) &= \log \binom{n}{\lfloor \rho n \rfloor} + n \log \nu \\
&= n \log n - \rho n \log(\rho n) - (1-\rho)n \log[(1-\rho)n] + n \log \nu + O(\log n) \\
&= n\left[g(\rho) + O\left(\frac{\log n}{n}\right)\right],
\end{aligned}$$

where the function $g : (0,1) \to \mathbb{R}$ is defined by

$$g(\rho) = \rho \log\left(\frac{1}{\rho}\right) + (1-\rho)\log\left(\frac{1}{1-\rho}\right) + \log \nu.$$

It is a continuous function of $\rho \in (0,1)$ and $\lim_{\rho \to 0} g(\rho) = \log \nu < 0$, hence there exists $\rho \in (0, 1/2)$ depending only on ν such that $g(\rho) \leq \frac{1}{2}\log \nu < 0$. For this value of ρ, we have

$$\log f_n(\lfloor \rho n \rfloor) \leq \frac{n}{2}\log \nu + O(\log n) \leq \frac{n}{4}\log \nu + O(1).$$

The result follows by combining the above with (3.54). ∎

Chapter 4

Stability of non-autonomous difference equations with applications to transport and wave propagation on networks

4.1 Introduction

As we have presented in Section 1.3, dynamics on networks has generated in the past decades an intense research activity within the PDE control community [6, 35, 63, 78, 90]. In particular, stability and stabilization of transport and wave propagation on networks raise challenging questions on the relationships between the asymptotic-in-time behavior of solutions on the one hand and, on the other hand, the topology of the network, its interconnection and damping laws at the vertices, and the rational dependence of the transit times on the network edges [2, 24, 47, 56, 171, 176]. A case of special interest is when some coefficients of the system are time-dependent and switch arbitrarily within a given set [7, 79, 149].

In this chapter, we address stability issues first for transport systems with time-dependent transmission conditions and then for wave propagation on networks with time-dependent damping terms. When the time-dependent coefficients switch arbitrarily in a given bounded set, we prove that the stability is robust with respect to variations of the lengths of the edges of the network preserving their rational dependence structure (see Corollary 4.48 for transport and Corollary 4.64 for wave propagation). Such robustness enables us to get the main result of the chapter, namely a necessary and sufficient criterion for exponential stability of wave propagation on networks: exponential stability holds for a network if and only if it is a tree and the damping is bounded away from zero at all external vertices but at most one (Theorem 4.65).

We address these issues by formulating them within the framework of non-autonomous linear difference equations

$$u(t) = \sum_{j=1}^{N} A_j(t) u(t - \Lambda_j), \qquad u(t) \in \mathbb{C}^d,\ (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N. \tag{4.1}$$

This standard approach relies on the d'Alembert decomposition and classical transformations of hyperbolic systems of PDEs into delay differential-difference equations [33, 54, 70, 106,133,160] (see also Example 1.42). Here, stability is meant uniformly with respect to the matrix-valued function $A(\cdot) = (A_1(\cdot), \dots, A_N(\cdot))$ belonging to a given class $\mathcal{A}$.

In the autonomous case, Equation (4.1) has a long history and its stability is completely characterized using spectral and Laplace transform techniques by the celebrated Hale–Silkowski criterion, recalled in Theorem 1.39 in Section 1.4.1. This criterion can also be used to evaluate the maximal Lyapunov exponent associated with $u(t) = \sum_{j=1}^{N} A_j u(t - \Lambda_j)$, i.e., the infimum over the exponential bounds for the corresponding semigroup. A remarkable feature of the Hale–Silkowski criterion is that, contrarily to the maximal Lyapunov exponent, it does not involve taking limits as time tends to infinity. An extension of these results has been obtained in [132] for the case where $\Lambda_1, \dots, \Lambda_N$ are not assumed to be rationally independent.

The non-autonomous case turns out to be more difficult since one does not have a general characterization of the exponential stability of (4.1) not involving limits as time tends to infinity. To illustrate that, consider the simple case $N = 1$ of a single delay and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ where $\mathfrak{B}$ is a bounded set of $d \times d$ matrices. Then the stability of (4.1) is equivalent to that of the discrete-time switched system $u_{n+1} = A_n u_n$ where $A_n \in \mathfrak{B}$, and it is characterized by the joint spectral radius of $\mathfrak{B}$ (see for instance [100, Section 2.2] and references therein) for which there is not yet a general characterization not involving limits as n tends to infinity.

Up to our knowledge, the only results regarding the stability of non-autonomous difference equations were obtained in [136], where sufficient conditions for stability are deduced from Perron–Frobenius Theorem. Our approach is rather based on a trajectory analysis relying on a suitable representation for solutions of (4.1), which expresses the solution $u(t)$ at time t as a linear combination of the initial condition u_0 evaluated at finitely many points identified explicitly. The matrix coefficients, denoted by Θ, are obtained in terms of the functions $A_1(\cdot), \dots, A_N(\cdot)$ and take into account the rational dependence structure of $\Lambda_1, \dots, \Lambda_N$ (Proposition 4.14). This representation provides a correspondence between the asymptotic behavior of solutions of (4.1), uniformly with respect to the initial condition u_0 and $A(\cdot) \in \mathcal{A}$, and that of the matrix coefficients Θ uniformly with respect to $A(\cdot) \in \mathcal{A}$ (Theorem 4.22). In the case where $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ for some bounded set $\mathfrak{B}$ of N-tuples of $d \times d$ matrices, we extend the results of [132], replacing ρ_{HS} in the Hale–Silkowski criterion, Theorem 1.39, by a generalization μ of the joint spectral radius. As a consequence of our analysis, and despite the lack of a closed and delay-independent formula for μ analogous to (1.51), we are able to show that stability for some N-tuple $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ is equivalent to stability for any choice of N-tuple $(L_1, \dots, L_N)$ having the same rational dependence structure as Λ (Corollaries 4.31 and 4.37).

The structure of the chapter goes as follows. Difference equations of the form (4.1) are discussed in Section 4.2. We start by establishing the well-posedness of the Cauchy problem and a representation formula for solutions in Sections 4.2.1 and 4.2.2. Stability criteria are given in Section 4.2.3 in terms of convergence of the coefficients and specified to the cases of shift-invariant classes $\mathcal{A}$ and arbitrary switching. In the latter case, we provide the above discussed generalization of the Hale–Silkowski criterion. Applications to transport equations are developed in Section 4.3 by exhibiting a correspondence with difference equations of the type (4.1). Thanks to the d'Alembert decomposition, results for transport equations are transposed to wave propagation on networks in Section 4.4. The topological characterization of exponential stability is given in Section 4.4.3.

Notations and definitions. All Banach and Hilbert spaces in this chapter are assumed to be complex.

A subset $\mathcal{A}$ of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ is said to be *uniformly locally bounded* if, for every compact time interval I, $\sup_{A\in\mathcal{A}}\|A\|_{L^\infty(I,\mathcal{M}_d(\mathbb{C})^N)}$ is finite. We say that $\mathcal{A}$ is *shift-invariant* if $A(\cdot+t)\in\mathcal{A}$ for every $A\in\mathcal{A}$ and $t\in\mathbb{R}$.

Throughout the chapter, we will use the indices δ, τ and ω in the notations of systems and functional spaces when dealing, respectively, with difference equations, transport systems and wave propagation.

4.2 Difference equations

Let $N,d\in\mathbb{N}^*$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(\mathbb{R}_+^*)^N$, and consider the system of time-dependent difference equations

$$\Sigma_\delta(\Lambda,A):\qquad u(t)=\sum_{j=1}^N A_j(t)u(t-\Lambda_j). \tag{4.2}$$

Here, $u(t)\in\mathbb{C}^d$ and $A=(A_1,\dots,A_N):\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$.

4.2.1 Well-posedness of the Cauchy problem

In this section, we show existence and uniqueness of solutions of the Cauchy problem associated with (4.2). We also consider the regularity of these solutions in terms of the initial condition and $A(\cdot)$.

Definition 4.1. Let $u_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$ and $A=(A_1,\dots,A_N):\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$. We say that $u:[-\Lambda_{\max},+\infty)\to\mathbb{C}^d$ is a *solution* of $\Sigma_\delta(\Lambda,A)$ with initial condition u_0 if it satisfies (4.2) for every $t\in\mathbb{R}_+$ and $u(t)=u_0(t)$ for $t\in[-\Lambda_{\max},0)$. In this case, we set, for $t\geq 0$, $u_t=u(\cdot+t)|_{[-\Lambda_{\max},0)}$.

We have the following result.

Proposition 4.2. *Let $u_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$ and $A=(A_1,\dots,A_N):\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$. Then $\Sigma_\delta(\Lambda,A)$ admits a unique solution $u:[-\Lambda_{\max},+\infty)\to\mathbb{C}^d$ with initial condition u_0.*

Proof. It suffices to build the solution u on $[-\Lambda_{\max},\Lambda_{\min})$ and then complete its construction on $[\Lambda_{\min},+\infty)$ by a standard inductive argument.

Suppose that $u:[-\Lambda_{\max},\Lambda_{\min})\to\mathbb{C}^d$ is a solution of $\Sigma_\delta(\Lambda,A)$ with initial condition u_0. Then, by (4.2), we have

$$u(t)=\begin{cases}\displaystyle\sum_{j=1}^N A_j(t)u_0(t-\Lambda_j), & \text{if } 0\leq t<\Lambda_{\min},\\ u_0(t), & \text{if } -\Lambda_{\max}\leq t<0.\end{cases} \tag{4.3}$$

Since the right-hand side is uniquely defined in terms of u_0 and A, we obtain the uniqueness of the solution. Conversely, if $u:[-\Lambda_{\max},\Lambda_{\min})\to\mathbb{C}^d$ is defined by (4.3), then (4.2) clearly holds for $t\in[-\Lambda_{\max},\Lambda_{\min})$ and thus u is a solution of $\Sigma_\delta(\Lambda,A)$. ■

Definition 4.3. For $p\in[1,+\infty]$, we use X_p^δ to denote the Banach space $\mathsf{X}_p^\delta=L^p([-\Lambda_{\max},0],\mathbb{C}^d)$ endowed with the usual L^p-norm denoted by $\|\cdot\|_p$.

Remark 4.4. If $u_0,v_0:[-\Lambda_{\max},0)\to\mathbb{C}^d$ are such that $u_0=v_0$ almost everywhere on $[-\Lambda_{\max},0)$ and $A,B:\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$ are such that $A=B$ almost everywhere on $\mathbb{R}_+$, then it follows from (4.3) that the solutions $u,v:[-\Lambda_{\max},+\infty)\to\mathbb{C}^d$ associated respectively with A, u_0 and

B, v_0 satisfy $u = v$ almost everywhere on $[-\Lambda_{\max}, +\infty)$. In particular, for initial conditions in X_p^δ, $p \in [1, +\infty]$, we still have existence and uniqueness of solutions, now in the sense of functions defined almost everywhere. If moreover $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, it easily follows from (4.3) that the corresponding solution u of $\Sigma_\delta(\Lambda, A)$ satisfies $u \in L^p_{\mathrm{loc}}([-\Lambda_{\max}, +\infty), \mathbb{C}^d)$.

Proposition 4.5. *Let $p \in [1, +\infty)$, $u_0 \in \mathsf{X}_p^\delta$, $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, and u be the solution of $\Sigma_\delta(\Lambda, A)$ with initial condition u_0. Then the X_p^δ-valued mapping $t \mapsto u_t$ defined on $\mathbb{R}_+$ is continuous.*

Proof. By Remark 4.4, $u_t \in \mathsf{X}_p^\delta$ for every $t \in \mathbb{R}_+$. Since $u_t(s) = u(s+t)$ for $s \in [-\Lambda_{\max}, 0)$, the continuity of $t \mapsto u_t$ follows from the continuity of translations in L^p (see, for instance, [152, Theorem 9.5]). ∎

Remark 4.6. The conclusion of Proposition 4.5 does not hold for $p = +\infty$ since translations in L^∞ are not continuous.

4.2.2 Representation formula for the solution

When $t \in [0, \Lambda_{\min})$, Equation (4.3) yields $u(t)$ in terms of the initial condition u_0. If $t \geq \Lambda_{\min}$, we use (4.2) to express the solution u at time t in terms of its values on previous times $t - \Lambda_j$, and, for each j such that $t > \Lambda_j$, we can reapply (4.2) at the time $t - \Lambda_j$ to obtain the expression of $u(t - \Lambda_j)$ in terms of u evaluated at previous times. By proceeding inductively, we can obtain an explicit expression for u in terms of u_0. For that purpose, let us introduce some notations.

Definition 4.7.

(a) An *increasing path* (in $\mathbb{N}^N$) is a finite sequence of points $(\mathbf{q}_k)_{k=1}^n$ in $\mathbb{N}^N$ such that, for $k \in [\![1, n-1]\!]$, $\mathbf{q}_{k+1}$ is obtained from $\mathbf{q}_k$ by adding 1 to exactly one of the coordinates of $\mathbf{q}_k$. For $n \in \mathbb{N}^*$ and $v = (v_1, \dots, v_n) \in [\![1, N]\!]^n$, we use $(\mathbf{p}_v(k))_{k=1}^{n+1}$ to denote the increasing path defined by

$$\mathbf{p}_v(k) = \sum_{j=1}^{k-1} e_{v_j}.$$

(b) For $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, we use $V_{\mathbf{n}}$ to denote the set

$$V_{\mathbf{n}} = \left\{(v_1, \dots, v_{|\mathbf{n}|_1}) \in [\![1, N]\!]^{|\mathbf{n}|_1} \;\middle|\; \mathbf{p}_v(|\mathbf{n}|_1 + 1) = \mathbf{n}\right\}, \tag{4.4}$$

i.e., $V_{\mathbf{n}}$ can be seen as the set of all increasing paths from 0 to $\mathbf{n}$.

(c) For $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$, $\mathbf{n} \in \mathbb{Z}^N$ and $t \in \mathbb{R}$, we define the matrix $\Xi_{\mathbf{n},t}^{\Lambda,A} \in \mathcal{M}_d(\mathbb{C})$ inductively by

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \begin{cases} 0, & \text{if } \mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ \mathrm{Id}_d, & \text{if } \mathbf{n} = 0, \\ \displaystyle\sum_{k=1}^N A_k(t) \Xi_{\mathbf{n}-e_k, t-\Lambda_k}^{\Lambda,A}, & \text{if } \mathbf{n} \in \mathbb{N}^N \setminus \{0\}. \end{cases} \tag{4.5}$$

We omit Λ, A or both from the notation $\Xi_{\mathbf{n},t}^{\Lambda,A}$ when they are clear from the context.

The following result provides a way to write $\Xi_{\mathbf{n},t}$ as a sum over $V_{\mathbf{n}}$ and as an alternative recursion formula.

Proposition 4.8. *For every* $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$ *and* $t \in \mathbb{R}$, *we have*

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}(t - \Lambda \cdot \mathbf{p}_v(k)) \tag{4.6}$$

and

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{k=1}^{N} \Xi_{\mathbf{n}-e_k,t}^{\Lambda,A} A_k(t - \Lambda \cdot \mathbf{n} + \Lambda_k). \tag{4.7}$$

Proof. We prove (4.6) by induction over $|\mathbf{n}|_1$. If $\mathbf{n} = e_i$ for some $i \in [\![1, N]\!]$, we have

$$\sum_{v \in V_{e_i}} \prod_{k=1}^{1} A_{v_k}(t) = A_i(t) = \Xi_{e_i,t}.$$

Let $R \in \mathbb{N}^*$ be such that (4.6) holds for every $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 = R$ and $t \in \mathbb{R}$. If $\mathbf{n} \in \mathbb{N}^N$ is such that $|\mathbf{n}|_1 = R + 1$ and $t \in \mathbb{R}$, we have, by (4.5) and the induction hypothesis, that

$$\begin{aligned}
\Xi_{\mathbf{n},t} &= \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} A_k(t) \Xi_{\mathbf{n}-e_k, t-\Lambda_k} = \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} \sum_{v \in V_{\mathbf{n}-e_k}} A_k(t) \prod_{r=1}^{|\mathbf{n}|_1 - 1} A_{v_r}(t - \Lambda_k - \Lambda \cdot \mathbf{p}_v(r)) \\
&= \sum_{v \in V_{\mathbf{n}}} \prod_{r=1}^{|\mathbf{n}|_1} A_{v_r}(t - \Lambda \cdot \mathbf{p}_v(r)),
\end{aligned}$$

where we use that $V_{\mathbf{n}} = \{(k, v) \mid k \in [\![1, N]\!],\ n_k \geq 1,\ v \in V_{\mathbf{n}-e_k}\}$ and that $e_k + \mathbf{p}_v(r) = \mathbf{p}_{(k,v)}(r+1)$. This establishes (4.6).

We now turn to the proof of (4.7). Since $\Xi_{e_j,t} = A_j(t)$, (4.7) is satisfied for $\mathbf{n} = e_j$, $j \in [\![1, N]\!]$. For $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 \geq 2$, the set $V_{\mathbf{n}}$ can be written as

$$V_{\mathbf{n}} = \{(v, k) \mid k \in [\![1, N]\!],\ n_k \geq 1,\ v \in V_{\mathbf{n}-e_k}\},$$

and thus, by (4.6), we have

$$\begin{aligned}
\Xi_{\mathbf{n},t} &= \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} \sum_{v \in V_{\mathbf{n}-e_k}} \left[\prod_{r=1}^{|\mathbf{n}|_1 - 1} A_{v_r}(t - \Lambda \cdot \mathbf{p}_v(r)) \right] A_k(t - \Lambda \cdot \mathbf{p}_v(|\mathbf{n}|_1)) \\
&= \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} \sum_{v \in V_{\mathbf{n}-e_k}} \left[\prod_{r=1}^{|\mathbf{n}|_1 - 1} A_{v_r}(t - \Lambda \cdot \mathbf{p}_v(r)) \right] A_k(t - \Lambda \cdot \mathbf{n} + \Lambda_k) = \sum_{k=1}^{N} \Xi_{\mathbf{n}-e_k,t} A_k(t - \Lambda \cdot \mathbf{n} + \Lambda_k). \qquad \blacksquare
\end{aligned}$$

In order to take into account the relations of rational dependence of $\Lambda_1, \ldots, \Lambda_N \in \mathbb{R}_+^*$ in the representation formula for the solution of $\Sigma_\delta(\Lambda, A)$, we set

$$\begin{aligned}
&Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\}, \\
V(\Lambda) &= \{L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L)\}, \qquad V_+(\Lambda) = V(\Lambda) \cap (\mathbb{R}_+^*)^N, \\
W(\Lambda) &= \{L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L)\}, \qquad W_+(\Lambda) = W(\Lambda) \cap (\mathbb{R}_+^*)^N.
\end{aligned} \tag{4.8}$$

Notice that $W(\Lambda) \subset V(\Lambda)$ and $W(\Lambda) = \{L \in V(\Lambda) \mid V(L) = V(\Lambda)\}$.

The point of view of this chapter is to prescribe $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$ and to describe the rational dependence structure of its components through the sets $Z(\Lambda)$, $V(\Lambda)$, and $W(\Lambda)$. Another possible viewpoint, which is the one used for instance in [132], is to fix $B \in \mathcal{M}_{N,h}(\mathbb{N})$ and consider the delays $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in \operatorname{Ran} B \cap (\mathbb{R}_+^*)^N$. We show in the next proposition that the two points of view are equivalent.

Proposition 4.9. *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$. There exist $h \in [\![1, N]\!]$, $\ell = (\ell_1, \dots, \ell_h) \in (\mathbb{R}_+^*)^h$ with rationally independent components, and $B \in \mathcal{M}_{N,h}(\mathbb{N})$ with $\operatorname{rk}(B) = h$ such that $\Lambda = B\ell$. Moreover, for every B as before, one has*

$$\begin{aligned} V(\Lambda) &= \operatorname{Ran} B, \\ W(\Lambda) &= \{B(\ell_1', \dots, \ell_h') \mid \ell_1', \dots, \ell_h' \text{ are rationally independent}\}. \end{aligned} \tag{4.9}$$

In particular, $W(\Lambda)$ is dense and of full measure in $V(\Lambda)$.

Proof. Let $\mathsf{V} = \operatorname{Span}_{\mathbb{Q}}\{\Lambda_1, \dots, \Lambda_N\}$, $h = \dim_{\mathbb{Q}} \mathsf{V}$, and $\{\lambda_1, \dots, \lambda_h\}$ be a basis of V with positive elements, so that $\Lambda = Au$ for some $A = (a_{ij}) \in \mathcal{M}_{N,h}(\mathbb{Q})$ and $u = (\lambda_1, \dots, \lambda_h) \in (\mathbb{R}_+^*)^h$. For $v \in \mathbb{R}^h \setminus \{0\}$, we denote by P_v the orthogonal projection in the direction of v, i.e., $P_v = v v^{\mathrm{T}} / \lvert v \rvert_2^2$.

Since $\mathbb{Q}^h$ is dense in $\mathbb{R}^h$, there exists a sequence of vectors $u_n = (r_{1,n}, \dots, r_{h,n})$ in $(\mathbb{Q}_+^*)^h$ converging to u as $n \to +\infty$, and we can further assume that the sequence is chosen in such a way that $\lvert P_{u_n} - P_u \rvert_2 \leq 1/n^2$ for every $n \in \mathbb{N}^*$.

For $n \in \mathbb{N}^*$, we define $T_n = P_{u_n} + \frac{1}{n}\left(\mathrm{Id}_h - P_{u_n}\right)$. This operator is invertible, with inverse $T_n^{-1} = P_{u_n} + n\left(\mathrm{Id}_h - P_{u_n}\right)$. Furthermore, both T_n and T_n^{-1} belong to $\mathcal{M}_h(\mathbb{Q})$. For $i \in [\![1, h]\!]$, we have

$$(T_n^{-1} e_i)^{\mathrm{T}} u = e_i^{\mathrm{T}} P_{u_n} u + n e_i^{\mathrm{T}} (\mathrm{Id}_h - P_{u_n}) u = e_i^{\mathrm{T}} P_{u_n} u + n e_i^{\mathrm{T}} (P_u - P_{u_n}) u$$

and thus $(T_n^{-1} e_i)^{\mathrm{T}} u \to e_i^{\mathrm{T}} u = \lambda_i$ as $n \to +\infty$. Since $\lambda_1, \dots, \lambda_h > 0$, there exists $n_0 \in \mathbb{N}^*$ such that

$$(T_n^{-1} e_i)^{\mathrm{T}} u > 0, \qquad \forall i \in [\![1, h]\!],\ \forall n \geq n_0. \tag{4.10}$$

For $i \in [\![1, N]\!]$, let $\alpha_i = (a_{ij})_{j \in [\![1, h]\!]} \in \mathbb{Q}^h$. For each $i \in [\![1, N]\!]$, we construct the sequence $(\alpha_{i,n})_{n \in \mathbb{N}^*}$ in $\mathbb{Q}^h$ by setting $\alpha_{i,n} = T_n \alpha_i$. It follows from the definition of T_n that $\alpha_{i,n}$ converges to $P_u \alpha_i = \frac{u u^{\mathrm{T}} \alpha_i}{\lvert u \rvert_2^2}$ as $n \to +\infty$. Since $u^{\mathrm{T}} \alpha_i = \sum_{j=1}^h a_{ij} \lambda_j = \Lambda_i > 0$ and the components of u are positive, we conclude that there exists $n_1 \geq n_0$ such that $\alpha_{i,n_1} \in (\mathbb{Q}_+)^h$ for every $i \in [\![1, N]\!]$.

Let $\ell = (T_{n_1}^{-1})^{\mathrm{T}} u$. By (4.10), $\ell_i = (T_{n_1}^{-1} e_i)^{\mathrm{T}} u > 0$ for every $i \in [\![1, h]\!]$. Since the components of u are rationally independent, $\ell_1, \dots, \ell_h$ are also rationally independent. Let $b_{ij} \in \mathbb{Q}_+$, $i \in [\![1, N]\!]$, $j \in [\![1, h]\!]$, be such that $\alpha_{i,n_1} = (b_{ij})_{j \in [\![1, h]\!]}$. Hence, for $i \in [\![1, N]\!]$,

$$\Lambda_i = u^{\mathrm{T}} \alpha_i = u^{\mathrm{T}} T_{n_1}^{-1} \alpha_{i,n_1} = \sum_{j=1}^h b_{ij} u^{\mathrm{T}} T_{n_1}^{-1} e_j = \sum_{j=1}^h b_{ij} \ell_j.$$

We then get the required result up to multiplying $B = (b_{ij})$ by a large integer and modifying ℓ in accordance.

We next prove that (4.9) holds for every B as before. (Our proof actually holds for every $B \in \mathcal{M}_{N,h}(\mathbb{Q})$ with $\operatorname{rk}(B) = h$ such that $\Lambda = B\ell$ for some $\ell \in (\mathbb{R}_+^*)^h$ with rationally independent components.) First notice that $Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \mathbf{n} \in \operatorname{Ker} B^{\mathrm{T}}\}$. Indeed, $\mathbf{n} \in Z(\Lambda)$ if and only if $\mathbf{n}$ is perpendicular in $\mathbb{R}^N$ to $B\ell$, which is equivalent to $\mathbf{n}^{\mathrm{T}} B = 0$ since $\ell_1, \dots, \ell_h$ are rationally independent. Moreover, remark that $\operatorname{Ker} B^{\mathrm{T}} = (\operatorname{Ran} B)^{\perp}$ admits a basis with integer coefficients since $\operatorname{Ran} B$ admits such a basis. To see that, it is enough to complete any basis of $\operatorname{Ran} B$ in

$\mathbb{Q}^N$ by $N-h$ vectors in $\mathbb{Q}^N$ and find a basis of $(\operatorname{Ran} B)^\perp$ by Gram–Schmidt orthogonalization. We finally deduce that $\operatorname{Span}_{\mathbb{R}}(Z(\Lambda)) = (\operatorname{Ran} B)^\perp$. Since by definition $V(\Lambda) = Z(\Lambda)^\perp$, we conclude that $V(\Lambda) = \operatorname{Ran} B$. As regards the characterization of $W(\Lambda)$, an argument goes as follows. Let $L \in V(\Lambda)$, so that $L = B\ell'$ for a certain $\ell' \in \mathbb{R}^h$. The components of ℓ' are rationally dependent if and only if $\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\{L_1, \dots, L_N\} < h$, i.e., $\dim_{\mathbb{R}} V(L) < \dim_{\mathbb{R}} V(\Lambda)$, which holds if and only if $L \notin W(\Lambda)$. ∎

We introduce the following additional definitions.

Definition 4.10. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$. We partition $[\![1, N]\!]$ and $\mathbb{Z}^N$ according to the equivalence relations $\sim$ and $\approx$ defined as follows: $i \sim j$ if $\Lambda_i = \Lambda_j$ and $\mathbf{n} \approx \mathbf{n}'$ if $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$. We use $[\cdot]$ to denote equivalence classes of both $\sim$ and $\approx$ and we set $\mathcal{J} = [\![1, N]\!]/\sim$ and $\mathcal{Z} = \mathbb{Z}^N/\approx$.

For $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $L \in V_+(\Lambda)$, $[\mathbf{n}] \in \mathcal{Z}$, $[i] \in \mathcal{J}$, and $t \in \mathbb{R}$, we define

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}',t}^{L,A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \sum_{j \in [i]} A_j(t), \tag{4.11}$$

and

$$\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j). \tag{4.12}$$

Remark 4.11. The expression for $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$ given in (4.11) is well-defined since, thanks to (4.5), all terms in the sum are equal to zero except finitely many. The expression for $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}$ given in (4.12) is also well-defined since, for every $L \in V_+(\Lambda)$, if $i \sim j$ and $\mathbf{n} \approx \mathbf{n}'$, one has $L_i = L_j$ and $L \cdot \mathbf{n} = L \cdot \mathbf{n}'$. In addition, notice that $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $[\mathbf{n}] \cap \mathbb{N}^N$ is nonempty, and, similarly, $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $[\mathbf{n}] \cap (\mathbb{N}^N \setminus \{0\})$ is nonempty. Another consequence of the above fact and (4.12) is that $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $t \geq 0$, since $[\mathbf{n} - e_j] \cap \mathbb{N}^N = \emptyset$ whenever $[\mathbf{n}] \in \mathcal{Z}$ and $[j] \in \mathcal{J}$ are such that $L \cdot \mathbf{n} - L_j < 0$.

Notice, moreover, that the matrices $\widehat{\Xi}$, $\widehat{A}$ and Θ depend on Λ only through $Z(\Lambda)$. Hence, if $\Lambda' \in W_+(\Lambda)$ (i.e., $Z(\Lambda') = Z(\Lambda)$), then

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda',A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \widehat{A}_{[i]}^{\Lambda'}(t), \qquad \Theta_{[\mathbf{n}],t}^{L,\Lambda,A} = \Theta_{[\mathbf{n}],t}^{L,\Lambda',A}.$$

From now on, we fix $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$ and our goal consists of deriving a suitable representation for the solutions of $\Sigma_\delta(L, A)$ for every $L \in V_+(\Lambda)$. Even though the above definitions depend on Λ, $L \in V_+(\Lambda)$ and A, we will sometimes omit (part of) this dependence from the notations when there is no risk of confusion.

Let us now provide further expressions for $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$.

Proposition 4.12. *For every $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$, we have*

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{[j] \in \mathcal{J}} \widehat{A}_{[j]}^{\Lambda}(t) \widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j}^{L,\Lambda,A}, \qquad \widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{[j] \in \mathcal{J}} \widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j), \tag{4.13}$$

and

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} \sum_{v \in V_{\mathbf{n}'}} \prod_{k=1}^{\lvert \mathbf{n}' \rvert_1} A_{v_k}(t - L \cdot \mathbf{p}_v(k)). \tag{4.14}$$

Proof. We have, by Definition 4.10 and Equation (4.5), that

$$\begin{aligned}
\widehat{\Xi}_{[\mathbf{n}],t} &= \sum_{\mathbf{n}'\in[\mathbf{n}]} \Xi_{\mathbf{n}',t} = \sum_{\mathbf{n}'\in[\mathbf{n}]} \sum_{j=1}^{N} A_j(t)\Xi_{\mathbf{n}'-e_j,t-L_j} = \sum_{j=1}^{N} A_j(t) \sum_{\mathbf{n}'\in[\mathbf{n}]} \Xi_{\mathbf{n}'-e_j,t-L_j} \\
&= \sum_{j=1}^{N} A_j(t) \sum_{\mathbf{m}\in[\mathbf{n}-e_j]} \Xi_{\mathbf{m},t-L_j} = \sum_{j=1}^{N} A_j(t)\widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j} \\
&= \sum_{[j]\in\mathcal{J}} \sum_{i\in[j]} A_i(t)\widehat{\Xi}_{[\mathbf{n}-e_i],t-L_i} = \sum_{[j]\in\mathcal{J}} \left(\sum_{i\in[j]} A_i(t)\right)\widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j} = \sum_{[j]\in\mathcal{J}} \widehat{A}_{[i]}(t)\widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j}.
\end{aligned}$$

The second expression is obtained similarly from Definition 4.10 and Equation (4.7) and the last one follows immediately from (4.6) and (4.11). ∎

Let us give a first representation for solutions of $\Sigma_\delta(L,A)$.

Lemma 4.13. *Let $L \in (\mathbb{R}_+^*)^N$, $A = (A_1,\dots,A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max},0) \to \mathbb{C}^d$. The corresponding solution $u : [-L_{\max},+\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L,A)$ is given for $t \geq 0$ by*

$$u(t) = \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -L_j\leq t-L\cdot\mathbf{n}<0}} \Xi^{L,A}_{\mathbf{n}-e_j,t} A_j(t-L\cdot\mathbf{n}+L_j)u_0(t-L\cdot\mathbf{n}). \tag{4.15}$$

Proof. Let $u : [-L_{\max},+\infty) \to \mathbb{C}^d$ be given for $t \geq 0$ by (4.15) and $u(t) = u_0(t)$ for $t \in [-L_{\max},0)$. Fix $t \geq 0$ and notice that

$$\begin{aligned}
&\sum_{j=1}^{N} A_j(t)u(t-L_j) \\
&= \sum_{\substack{j=1 \\ t\geq L_j}}^{N} \sum_{\substack{(\mathbf{n},k)\in\mathbb{N}^N\times[\![1,N]\!] \\ -L_k\leq t-L_j-L\cdot\mathbf{n}<0 \\ n_k\geq 1}} A_j(t)\Xi^{L,A}_{\mathbf{n}-e_k,t-L_j} A_k(t-L_j-L\cdot\mathbf{n}+L_k)u_0(t-L_j-L\cdot\mathbf{n}) \\
&\quad + \sum_{\substack{j=1 \\ t<L_j}}^{N} A_j(t)u_0(t-L_j).
\end{aligned} \tag{4.16}$$

Consider the sets

$$\begin{gathered}
B_1(t) = \{(\mathbf{n},k,j) \in \mathbb{N}^N \times [\![1,N]\!]^2 \mid t \geq L_j,\ -L_k \leq t-L_j-L\cdot\mathbf{n} < 0,\ n_k \geq 1\}, \\
B_2(t) = \{j \in [\![1,N]\!] \mid t < L_j\}, \\
C_1(t) = \{(\mathbf{n},k,j) \in \mathbb{N}^N \times [\![1,N]\!]^2 \mid -L_k \leq t-L\cdot\mathbf{n} < 0,\ n_k \geq 1,\ n_j \geq 1+\delta_{jk},\ \mathbf{n} \neq e_k\}, \\
C_2(t) = \{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \mid -L_k \leq t-L\cdot\mathbf{n} < 0,\ \mathbf{n} = e_k\},
\end{gathered}$$

and the functions $\varphi_i : B_i(t) \to C_i(t)$, $i \in \{1,2\}$, given by

$$\varphi_1(\mathbf{n},k,j) = (\mathbf{n}+e_j,k,j), \qquad \varphi_2(j) = (e_j,j).$$

One can check that φ_1 and φ_2 are well-defined and injective. We claim that they are also bijective. For the surjectivity of φ_1, we take $(\mathbf{n},k,j) \in C_1(t)$ and set $\mathbf{m} = \mathbf{n}-e_j$. Since $n_j \geq 1$, one has $\mathbf{m} \in \mathbb{N}^N$. Since $n_k \geq 1$, $n_j \geq 1 + \delta_{jk}$, one has $t \geq L \cdot \mathbf{n} - L_k \geq L_j + L_k - L_k = L_j$. The inequalities $-L_k \leq t - L_j - L \cdot \mathbf{m} < 0$ and $n_k \geq 1$ are trivially satisfied, and thus $(\mathbf{m}, k, j) \in B_1(t)$, which shows the surjectivity of φ_1 since one clearly has $\varphi_1(\mathbf{m}, k, j) = (\mathbf{n}, k, j)$. For the surjectivity of φ_2, we take $(\mathbf{n}, k) \in C_2(t)$, which then satisfies $\mathbf{n} = e_k$ and $t < L \cdot \mathbf{n} = L_k$. This shows that $k \in B_2(t)$ and, since $\varphi_2(k) = (\mathbf{n}, k)$, we obtain that φ_2 is surjective.

Thanks to the bijections φ_1, φ_2, and (4.5), (4.16) becomes

$$
\begin{aligned}
\sum_{j=1}^{N} A_j(t) u(t - L_j) &= \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L \cdot \mathbf{n} < 0 \\ n_k \geq 1,\, \mathbf{n} \neq e_k}} \sum_{\substack{j=1 \\ n_j \geq 1 + \delta_{jk}}}^{N} A_j(t) \Xi^{L,A}_{\mathbf{n}-e_k-e_j, t-L_j} A_k(t - L \cdot \mathbf{n} + L_k) u_0(t - L \cdot \mathbf{n}) \\
&\quad + \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L \cdot \mathbf{n} < 0, \\ \mathbf{n} = e_k}} A_k(t - L \cdot \mathbf{n} + L_k) u_0(t - L \cdot \mathbf{n}) \\
&= \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L \cdot \mathbf{n} < 0 \\ n_k \geq 1,\, \mathbf{n} \neq e_k}} \Xi^{L,A}_{\mathbf{n}-e_k, t} A_k(t - L \cdot \mathbf{n} + L_k) u_0(t - L \cdot \mathbf{n}) \\
&\quad + \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L \cdot \mathbf{n} < 0, \\ \mathbf{n} = e_k}} \Xi^{L,A}_{0,t} A_k(t - L \cdot \mathbf{n} + L_k) u_0(t - L \cdot \mathbf{n}) \\
&= \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L \cdot \mathbf{n} < 0}} \Xi^{L,A}_{\mathbf{n}-e_k, t} A_k(t - L \cdot \mathbf{n} + L_k) u_0(t - L \cdot \mathbf{n}) = u(t),
\end{aligned}
$$

which shows that u satisfies (4.2). ■

We can now give the main result of this section.

Proposition 4.14. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The corresponding solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is given for $t \geq 0$ by*

$$
u(t) = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z} \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \Theta^{L,\Lambda,A}_{[\mathbf{n}],t} u_0(t - L \cdot \mathbf{n}), \tag{4.17}
$$

where the coefficients Θ are defined in (4.12).

Proof. Equation (4.17) follows immediately from (4.15) and from the fact that the function $\varphi : \mathbb{N}^N \times [\![1,N]\!] \to \mathcal{Z} \times \mathbb{N}^N \times \mathcal{J} \times [\![1,N]\!]$ given by $\varphi(\mathbf{n}, j) = ([\mathbf{n}], \mathbf{n}, [j], j)$ is a bijective map from $\{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1,N]\!] \mid -L_j \leq t - L \cdot \mathbf{n} < 0\}$ to $\{([\mathbf{m}], \mathbf{n}, [i], j) \in \mathcal{Z} \times \mathbb{N}^N \times \mathcal{J} \times [\![1,N]\!] \mid \mathbf{n} \in [\mathbf{m}],\ j \in [i],\ t < L \cdot \mathbf{n} \leq t + L_{\max},\ L \cdot \mathbf{n} - L_j \leq t\}$ for every $t \in \mathbb{R}$. ■

Remark 4.15. Using the link between transport and difference equations highlighted in Section 1.3, it follows that Lemma 4.13 and Proposition 4.14 generalize Theorems 3.15 and 3.18.

4.2.3 Asymptotic behavior of solutions in terms of the coefficients

Let us fix a matrix norm $\lvert\cdot\rvert$ on $\mathcal{M}_d(\mathbb{C})$ in the whole section. Let $C_1, C_2 > 0$ be such that

$$
C_1 \lvert A \rvert_p \leq \lvert A \rvert \leq C_2 \lvert A \rvert_p, \qquad \forall A \in \mathcal{M}_d(\mathbb{C}),\ \forall p \in [1, +\infty]. \tag{4.18}
$$

Let $\mathcal{A}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$. The family of all systems $\Sigma_\delta(L, A)$ for $A \in \mathcal{A}$ is denoted by $\Sigma_\delta(L, \mathcal{A})$. We wish to characterize the asymptotic behavior of $\Sigma_\delta(L, \mathcal{A})$ (i.e., uniformly with respect to $A \in \mathcal{A}$) in terms of the behavior of the coefficients $\widehat{\Xi}_{[\mathbf{n}],t}$ and $\Theta_{[\mathbf{n}],t}$. For that purpose, we introduce the following definitions.

Definition 4.16. Let $L \in (\mathbb{R}_+^*)^N$.

(a) For $p \in [1, +\infty]$, we say that $\Sigma_\delta(L, \mathcal{A})$ is of *exponential type* $\gamma \in \mathbb{R}$ in X_p^δ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$ and $u_0 \in \mathsf{X}_p^\delta$, the corresponding solution u of $\Sigma_\delta(L, A)$ satisfies, for every $t \geq 0$,

$$\|u_t\|_p \leq K e^{(\gamma+\varepsilon)t} \|u_0\|_p.$$

We say that $\Sigma_\delta(L, \mathcal{A})$ is *exponentially stable* in X_p^δ if it is of negative exponential type.

(b) Let $\Lambda \in (\mathbb{R}_+^*)^N$ be such that $L \in V_+(\Lambda)$. We say that $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-*exponential type* $\gamma \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, we have

$$\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq K e^{(\gamma+\varepsilon)t}.$$

(c) Let $\Lambda \in (\mathbb{R}_+^*)^N$ be such that $L \in V_+(\Lambda)$. We say that $\Sigma_\delta(L, \mathcal{A})$ is of $(\widehat{\Xi}, \Lambda)$-*exponential type* $\gamma \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in \mathbb{R}$, we have

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq K e^{(\gamma+\varepsilon)L\cdot\mathbf{n}}.$$

(d) For $p \in [1, +\infty]$, the *maximal Lyapunov exponent of* $\Sigma_\delta(L, \mathcal{A})$ in X_p^δ is defined as

$$\lambda_p(L, \mathcal{A}) = \limsup_{t\to+\infty} \sup_{A\in\mathcal{A}} \sup_{\substack{u_0 \in \mathsf{X}_p^\delta \\ \|u_0\|_p = 1}} \frac{\log\|u_t\|_p}{t},$$

where u denotes the solution of $\Sigma_\delta(L, A)$ with initial condition u_0.

Remark 4.17. Let $L \in (\mathbb{R}_+^*)^N$ and $\mu \in \mathbb{R}$. For every $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$ and u solution of $\Sigma_\delta(L, A)$, it follows from (4.2) that $t \mapsto e^{\mu t} u(t)$ is a solution of the system $\Sigma_\delta(L, (e^{\mu L_1} A_1, \dots, e^{\mu L_N} A_N))$. As a consequence, if $\mathcal{A} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ and

$$\mathcal{A}_\mu = \{(e^{\mu L_1} A_1, \dots, e^{\mu L_N} A_N) \mid A = (A_1, \dots, A_N) \in \mathcal{A}\},$$

one has $\lambda_p(L, \mathcal{A}_\mu) = \lambda_p(L, \mathcal{A}) + \mu$.

The link between exponential type and maximal Lyapunov exponent of $\Sigma_\delta(L, \mathcal{A})$ is provided by the following proposition.

Proposition 4.18. *Let $L \in (\mathbb{R}_+^*)^N$, $\mathcal{A}$ be uniformly locally bounded, and $p \in [1, +\infty]$. Then*

$$\lambda_p(L, \mathcal{A}) = \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L, \mathcal{A}) \text{ is of exponential type } \gamma \text{ in } \mathsf{X}_p^\delta\}.$$

In particular, $\Sigma_\delta(L, \mathcal{A})$ is exponentially stable if and only if $\lambda_p(L, \mathcal{A}) < 0$.

Proof. Let $\gamma \in \mathbb{R}$ be such that $\Sigma_\delta(L,\mathcal{A})$ is of exponential type γ in X_p^δ. It is clear from the definition that $\lambda_p(L,\mathcal{A}) \leq \gamma$. We are left to prove that $\Sigma_\delta(L,\mathcal{A})$ is of exponential type $\lambda_p(L,\mathcal{A})$ when the latter is finite. Let $\varepsilon > 0$. From the definition of $\lambda_p(L,\mathcal{A})$, there exists $t_0 > 0$ such that, for every $t \geq t_0$, $A \in \mathcal{A}$, and $u_0 \in \mathsf{X}_p^\delta$, one has

$$\|u_t\|_p \leq e^{(\lambda_p(L,\mathcal{A})+\varepsilon)t} \|u_0\|_p.$$

Since $\mathcal{A}$ is uniformly locally bounded, by using (4.15) and (4.6), one deduces that there exists $K > 0$ such that, for every $t \in [0,t_0]$, $A \in \mathcal{A}$, and $u_0 \in \mathsf{X}_p^\delta$, one has $\|u_t\|_p \leq K\|u_0\|_p$. Hence the conclusion. ■

Remark 4.19. Similarly, one proves that, for $\Lambda \in (\mathbb{R}_+^*)^N$ and $L \in V_+(\Lambda)$,

$$\limsup_{L\cdot\mathbf{n}\to+\infty} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in(L\cdot\mathbf{n}-L_{\max},L\cdot\mathbf{n})} \frac{\log\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{t} = \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L,\mathcal{A}) \text{ is of } (\Theta,\Lambda)\text{-exponential type } \gamma\}$$

and

$$\limsup_{L\cdot\mathbf{n}\to+\infty} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{\log\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{L\cdot\mathbf{n}} = \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L,\mathcal{A}) \text{ is of } (\widehat{\Xi},\Lambda)\text{-exponential type } \gamma\}.$$

4.2.3.1 General case

The following result, which is a generalization of Proposition 3.24, uses the representation formula (4.17) for the solutions of $\Sigma_\delta(L,A)$ in order to provide upper bounds on their growth.

Proposition 4.20. *Let $L \in V_+(\Lambda)$. Suppose that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}_+^*$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L\cdot\mathbf{n} - L_{\max}, L\cdot\mathbf{n})$, one has*

$$\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq f(t). \tag{4.19}$$

Then there exists $C > 0$ such that, for every $A \in \mathcal{A}$, $p \in [1,+\infty]$, and $u_0 \in \mathsf{X}_p^\delta$, the corresponding solution u of $\Sigma_\delta(L,A)$ satisfies, for every $t \geq 0$,

$$\|u_t\|_p \leq C(t+1)^{N-1} \max_{s\in[t-L_{\max},t]} f(s)\|u_0\|_p. \tag{4.20}$$

Proof. Let $A \in \mathcal{A}$, $p \in [1,+\infty)$, $u_0 \in \mathsf{X}_p^\delta$, and u be the solution of $\Sigma_\delta(L,A)$ with initial condition u_0. For $t \in \mathbb{R}_+$, we write $\mathcal{Y}_t = \{[\mathbf{n}] \in \mathcal{Z} \mid t < L\cdot\mathbf{n} \leq t + L_{\max},\ [\mathbf{n}] \cap \mathbb{N}^N \neq \emptyset\}$ and $Y_t = \#\mathcal{Y}_t$. Thanks to Proposition 4.14, Remark 4.11, and (4.19), we have, for $t \geq L_{\max}$,

$$\begin{aligned}
\|u_t\|_p^p &= \int_{t-L_{\max}}^t \left|\sum_{[\mathbf{n}]\in\mathcal{Y}_s} \Theta_{[\mathbf{n}],s} u_0(s-L\cdot\mathbf{n})\right|_p^p ds \leq \int_{t-L_{\max}}^t Y_s^{p-1} \sum_{[\mathbf{n}]\in\mathcal{Y}_s} \left|\Theta_{[\mathbf{n}],s} u_0(s-L\cdot\mathbf{n})\right|_p^p ds \\
&\leq C_1^{-p} \int_{t-L_{\max}}^t Y_s^{p-1} f(s)^p \sum_{[\mathbf{n}]\in\mathcal{Y}_s} |u_0(s-L\cdot\mathbf{n})|_p^p\, ds \\
&\leq C_1^{-p} \max_{s\in[t-L_{\max},t]} f(s)^p \int_{t-L_{\max}}^t Y_s^{p-1} \sum_{[\mathbf{n}]\in\mathcal{Y}_s} |u_0(s-L\cdot\mathbf{n})|_p^p\, ds.
\end{aligned}$$

We clearly have $Y_t \leq \#\{\mathbf{n} \in \mathbb{N}^N \mid t < L \cdot \mathbf{n} \leq t + L_{\max}\}$. For $\mathbf{n} \in \mathbb{N}^N$, we denote $\mathfrak{C}_{\mathbf{n}} = \{x \in \mathbb{R}^N \mid n_i < x_i < n_i + 1 \text{ for every } i \in [\![1, N]\!]\}$. This defines a family of pairwise disjoint open hypercubes of unit volume. Thus

$$Y_t \leq \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ t<L\cdot\mathbf{n}\leq t+L_{\max}}} \operatorname{Vol} \mathfrak{C}_{\mathbf{n}} = \operatorname{Vol}\left(\bigcup_{\substack{\mathbf{n}\in\mathbb{N}^N \\ t<L\cdot\mathbf{n}\leq t+L_{\max}}} \mathfrak{C}_{\mathbf{n}}\right) \leq \operatorname{Vol}\{x \in (\mathbb{R}_+)^N \mid t < L \cdot x < t + \lvert L\rvert_1 + L_{\max}\}.$$

Then there exists $C_3 > 0$ only depending on L and N such that $Y_t \leq C_3(t+1)^{N-1}$. Thus,

$$\begin{aligned}\lVert u_t\rVert_p^p &\leq C_1^{-p} C_3^{p-1} (t+1)^{(N-1)(p-1)} \max_{s\in[t-L_{\max},t]} f(s)^p \int_{t-L_{\max}}^{t} \sum_{[\mathbf{n}]\in\mathcal{Y}_s} \lvert u_0(s - L\cdot\mathbf{n})\rvert_p^p \, ds \\ &= C_1^{-p} C_3^{p-1} (t+1)^{(N-1)(p-1)} \max_{s\in[t-L_{\max},t]} f(s)^p \int_{-L_{\max}}^{0} \sum_{[\mathbf{n}]\in\mathcal{Y}_{t-L_{\max}-s}} \lvert u_0(s)\rvert_p^p \, ds.\end{aligned}$$

Similarly, there exists $C_4 > 0$ only depending on L and N such that, for every $t \in \mathbb{R}_+$ and $s \in [-L_{\max}, 0]$, $Y_{t-L_{\max}-s} \leq C_4(t+1)^{N-1}$, yielding (4.20) for $t \geq L_{\max}$. One can easily show that, for $0 \leq t \leq L_{\max}$, we have $\lVert u_t\rVert_p \leq C' \lVert u_0\rVert_p$ for some constant C' independent of p and u_0, and so (4.20) holds for every $t \geq 0$. The case $p = +\infty$ is treated by similar arguments. ∎

When $L \in W_+(\Lambda)$, we also have the following lower bound for solutions of $\Sigma_\delta(L, \mathcal{A})$.

Proposition 4.21. *Let $L \in W_+(\Lambda)$ and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Suppose that there exist $A \in \mathcal{A}$, $\mathbf{n}_0 \in \mathbb{N}^N$, and a set of positive measure $S \subset (L \cdot \mathbf{n}_0 - L_{\max}, L \cdot \mathbf{n}_0)$ such that, for every $s \in S$,*

$$\left\lvert \Theta_{[\mathbf{n}_0],s}^{L,\Lambda,A} \right\rvert > f(s). \tag{4.21}$$

Then there exist a constant $C > 0$ independent of f, an initial condition $u_0 \in L^\infty([-L_{\max}, 0], \mathbb{C}^d)$, and $t > 0$, such that, for every $p \in [1, +\infty]$, the solution u of $\Sigma_\delta(L, A)$ with initial condition u_0 satisfies

$$\lVert u_t\rVert_p > C \min_{s\in[t-L_{\max},t]} f(s) \lVert u_0\rVert_p.$$

Proof. According to Remark 4.11, one has $\Theta_{[\mathbf{n}],s}^{L,\Lambda,A} = \Theta_{[\mathbf{n}],s}^{L,L,A}$ for every $[\mathbf{n}] \in \mathcal{Z}$ and $s \in \mathbb{R}$, and therefore we assume for the rest of the argument that $\Lambda = L$ and we drop the upper index L, L, A.

For $s \in S$, one has $\left\lvert \Theta_{[\mathbf{n}_0],s} \right\rvert_\infty > C_2^{-1} f(s)$, where C_2 is defined in (4.18). Using (4.21) and Remark 4.11, one derives that $S \subset [0, +\infty)$.

For every $s \in S$, one has

$$C_2^{-1} f(s) < \left\lvert \Theta_{[\mathbf{n}_0],s} \right\rvert_\infty \leq \sum_{j=1}^{d} \left\lvert \Theta_{[\mathbf{n}_0],s} e_j \right\rvert_\infty,$$

and thus there exist $j_0 \in [\![1, d]\!]$ and a subset $\tilde{S} \subset S$ of positive measure such that, for every $s \in \tilde{S}$ and $p \in [1, +\infty]$, one has

$$C_2^{-1} d^{-1} f(s) < \left\lvert \Theta_{[\mathbf{n}_0],s} e_{j_0} \right\rvert_\infty \leq \left\lvert \Theta_{[\mathbf{n}_0],s} e_{j_0} \right\rvert_p. \tag{4.22}$$

In order to simplify the notations in the sequel, we write S instead of $\tilde{S}$.

Let $t_0 \in S \setminus \{0\}$ be such that, for every $\varepsilon > 0$, $(t_0 - \varepsilon, t_0 + \varepsilon) \cap S$ has positive measure. Let $\delta > 0$ be such that

$$2\delta < \min\left\{2t_0, L\cdot\mathbf{n}_0 - t_0, t_0 - L\cdot\mathbf{n}_0 + L_{\max}, \min_{\substack{\mathbf{n}\in\mathbb{N}^N\\ L\cdot(\mathbf{n}-\mathbf{n}_0)\neq 0}} \left|L\cdot(\mathbf{n}-\mathbf{n}_0)\right|\right\}.$$

Such a choice is possible since $t_0 \in (L\cdot\mathbf{n}_0 - L_{\max}, L\cdot\mathbf{n}_0)$, $t_0 \in S\setminus\{0\} \subset \mathbb{R}_+^*$, and $\{L\cdot\mathbf{n} \mid \mathbf{n}\in\mathbb{N}^N\}$ is locally finite.

Let $S_1 = (S - t_0) \cap (-\delta, \delta)$, which is, by construction, of positive measure, and $\mu : \mathbb{R} \to \mathbb{R}$ be any non-zero bounded measurable function which is zero outside S_1. Define $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$ by

$$u_0(s) = \mu(s - t_0 + L\cdot\mathbf{n}_0)e_{j_0}$$

and let u be the solution of $\Sigma_\delta(L, A)$ with initial condition u_0. For $s \in (-\delta, \delta)$, we have $t_0 + s > 0$ since $t_0 > \delta$. By Proposition 4.14, one has

$$u(t_0 + s) = \sum_{\substack{[\mathbf{n}]\in\mathcal{Z}\\ t_0+s<L\cdot\mathbf{n}\leq t_0+s+L_{\max}}} \Theta_{[\mathbf{n}],t_0+s}\mu(s + L\cdot(\mathbf{n}_0 - \mathbf{n}))e_{j_0}. \tag{4.23}$$

If $L\cdot\mathbf{n} \neq L\cdot\mathbf{n}_0$, we have $|L\cdot(\mathbf{n}-\mathbf{n}_0)| > 2\delta$, and so $|s + L\cdot(\mathbf{n}_0 - \mathbf{n})| > \delta$, which shows that $\mu(s + L\cdot(\mathbf{n}_0 - \mathbf{n})) = 0$. Hence, Equation (4.23) reduces to $u(t_0+s) = \Theta_{[\mathbf{n}_0],t_0+s}\mu(s)e_{j_0}$. We finally obtain, using (4.22) and letting $t = t_0 + \delta$, that, for $p \in [1, +\infty)$,

$$\begin{aligned} \|u_t\|_p^p \geq \left\|u_{t_0}\right\|_{L^p([-\delta,\delta],\mathbb{C}^d)}^p \geq \int_{S_1} |u(t_0+s)|_p^p\,ds = \int_{S_1} \left|\Theta_{[\mathbf{n}_0],t_0+s}e_{j_0}\right|_p^p \left|\mu(s)\right|^p ds \\ > C_2^{-p}d^{-p}\int_{S_1} f(t_0+s)^p\left|\mu(s)\right|^p ds \geq C_2^{-p}d^{-p}\min_{s\in[t-L_{\max},t]} f(s)^p\|u_0\|_p^p. \end{aligned} \tag{4.24}$$

A similar estimate holds in the case $p = +\infty$, yielding the result. ∎

As a corollary of Propositions 4.20 and 4.21, by taking f of the type $f(t) = Ke^{(\gamma+\varepsilon)t}$, one obtains the following theorem. The last equality follows from Proposition 4.18 and Remark 4.19.

Theorem 4.22. *Let $\Lambda \in (\mathbb{R}_+^*)^N$ and $\mathcal{A}$ be uniformly locally bounded. For every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ then, for every $p \in [1, +\infty]$, it is of exponential type γ in X_p^δ. Conversely, for every $L \in W_+(\Lambda)$, if there exists $p \in [1, +\infty]$ such that $\Sigma_\delta(L, \mathcal{A})$ is of exponential type γ in X_p^δ, then it is of (Θ, Λ)-exponential type γ. Finally, for every $L \in W_+(\Lambda)$ and $p \in [1, +\infty]$,*

$$\lambda_p(L, \mathcal{A}) = \limsup_{L\cdot\mathbf{n}\to+\infty}\sup_{A\in\mathcal{A}}\operatorname*{ess\,sup}_{t\in(L\cdot\mathbf{n}-L_{\max},L\cdot\mathbf{n})} \frac{\log\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{t}. \tag{4.25}$$

Remark 4.23. It also follows from Proposition 4.20 that, in the first part of the theorem, the constant $K > 0$ in the definition of exponential type of $\Sigma_\delta(L, \mathcal{A})$ can be chosen independently of $p \in [1, +\infty]$. Moreover, the left-hand side of (4.25) does not depend on p and its right-hand side does not depend on Λ.

4.2.3.2 Shift-invariant classes

We start this section by the following technical result.

Lemma 4.24. *For every $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\mathbf{n} \in \mathbb{Z}^N$, and $t, \tau \in \mathbb{R}$, we have*

$$\Xi_{\mathbf{n},t+\tau}^{L,A} = \Xi_{\mathbf{n},t}^{L,A(\cdot+\tau)} \qquad \text{and} \qquad \widehat{\Xi}_{[\mathbf{n}],t+\tau}^{L,\Lambda,A} = \widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A(\cdot+\tau)}.$$

Proof. The first part holds trivially if $\mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N$ or if $\mathbf{n} = 0$, for, in these cases, it follows from (4.5) that $\Xi_{\mathbf{n},t}^{L,A}$ does not depend on t and A. If $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, the conclusion follows as a consequence of the explicit formula (4.6) for $\Xi_{\mathbf{n},t}^{L,A}$. The second part is a consequence of the first and (4.11). ∎

We next provide a proposition establishing a relation between the behavior of $\widehat{\Xi}_{[\mathbf{n}],t}$ and $\Theta_{[\mathbf{n}],t}$. Notice that, if a subset $\mathcal{A}$ of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ is shift-invariant, then $\mathcal{A}$ is uniformly locally bounded if and only if it is bounded.

Proposition 4.25. *Let $\mathcal{A}$ be a bounded shift-invariant subset of $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, $L \in V_+(\Lambda)$, and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Then the following assertions hold.*

(a) *If $\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq f(t)$ for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, then, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and almost every $t \in \mathbb{R}$, one has $\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq \max_{s \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}]} f(s)$.*

(b) *If $\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq f(L \cdot \mathbf{n})$ for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in \mathbb{R}$, then there exists a constant $C > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, one has $\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq C \max_{s \in [t - L_{\max}, t + L_{\max}]} f(s)$.*

Proof. We start by showing (a). Let $A \in \mathcal{A}$ and $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$. For every $k \in \mathbb{Z}$, there exists a set $N_k \subset [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n})$ of measure zero such that, for every $t \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}) \setminus N_k$,

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A(\cdot-kL_{\min})}\right| = \left|\sum_{[j] \in \mathcal{J}} \widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A(\cdot-kL_{\min})} \widehat{A}_{[j]}^{\Lambda}(t - kL_{\min} - L \cdot \mathbf{n} + L_j)\right| = \left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A(\cdot-kL_{\min})}\right| \leq f(t),$$

where we use Proposition 4.12, the fact that $L \cdot \mathbf{n} - L_j \leq L \cdot \mathbf{n} - L_{\min} \leq t$ for every $[j] \in \mathcal{J}$, and Equation (4.12).

Let $N = \bigcup_{k \in \mathbb{Z}} (N_k - kL_{\min})$, which is of measure zero. For $t \in \mathbb{R} \setminus N$, let $k \in \mathbb{Z}$ be such that $t \in [L \cdot \mathbf{n} - (k+1)L_{\min}, L \cdot \mathbf{n} - kL_{\min})$, so that $t + kL_{\min} \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n})$. Since $t \notin N$, we have $t + kL_{\min} \notin N_k$, and so, using Lemma 4.24, we obtain that

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| = \left|\widehat{\Xi}_{[\mathbf{n}],t+kL_{\min}}^{L,\Lambda,A(\cdot-kL_{\min})}\right| \leq f(t + kL_{\min}) \leq \max_{s \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}]} f(s).$$

Let us now show (b). Without loss of generality, the norm $|\cdot|$ is sub-multiplicative. Since $\mathcal{A}$ is bounded, there exists $M > 0$ such that, for every $A \in \mathcal{A}$, $j \in \llbracket 1, N \rrbracket$, and $t \in \mathbb{R}$, we have $\left|A_j(t)\right| \leq M$. Let $A \in \mathcal{A}$. For every $\mathbf{n} \in \mathbb{N}^N$, let $N_{[\mathbf{n}]}$ be a set of measure zero such that $\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq f(L \cdot \mathbf{n})$ holds for every $t \in \mathbb{R} \setminus N_{[\mathbf{n}]}$. Let $N = \bigcup_{\mathbf{n} \in \mathbb{N}^N} N_{[\mathbf{n}]}$, which is of measure zero. If $\mathbf{n} \in \mathbb{N}^N$ and $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n}) \setminus N$, then

$$\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \left|\widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A}\right| \left|\widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j)\right| \leq NM \sum_{[j] \in \mathcal{J}} f(L \cdot \mathbf{n} - L_j) \leq C \max_{s \in [t - L_{\max}, t + L_{\max}]} f(s),$$

where $C = N^2 M$. ∎

As an immediate consequence of the previous proposition and Theorem 4.22, we have the following theorem, which improves Theorem 4.22 by replacing (Θ,Λ)-exponential type by $(\widehat{\Xi},\Lambda)$-exponential type.

Theorem 4.26. *Let $\Lambda \in (\mathbb{R}_+^*)^N$ and $\mathcal{A}$ be a bounded shift-invariant subset of $L^\infty(\mathbb{R},\mathcal{M}_d(\mathbb{C})^N)$. For every $L \in V_+(\Lambda)$, $\Sigma_\delta(L,\mathcal{A})$ is of $(\widehat{\Xi},\Lambda)$-exponential type γ if and only if it is of (Θ,Λ)-exponential type γ.*

As a consequence, for every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L,\mathcal{A})$ is of $(\widehat{\Xi},\Lambda)$-exponential type γ then, for every $p \in [1,+\infty]$, it is of exponential type γ in X_p^δ. Conversely, for every $L \in W_+(\Lambda)$, if there exists $p \in [1,+\infty]$ such that $\Sigma_\delta(L,\mathcal{A})$ is of exponential type γ in X_p^δ, then it is of $(\widehat{\Xi},\Lambda)$-exponential type γ. Finally, for every $L \in W_+(\Lambda)$ and $p \in [1,+\infty]$,

$$\lambda_p(L,\mathcal{A}) = \limsup_{L\cdot\mathbf{n}\to+\infty} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{\log\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{L\cdot\mathbf{n}}. \tag{4.26}$$

Remark 4.27. Theorem 4.26 improves Theorem 4.22 in the sense that the coefficients $\Xi_{\mathbf{n},t}^{L,A}$ and $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$ are in general easier to compute or estimate than $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}$ thanks to the recursion formulas (4.5), (4.7), and (4.13). Notice that, using the link between transport and difference equations highlighted in Section 1.3, the estimates carried out in Sections 3.4.2, 3.4.3, and 3.4.4 correspond to estimates on the coefficients $\Xi_{\mathbf{n},t}^{L,A}$.

4.2.3.3 Arbitrary switching

We consider in this section $\mathcal{A}$ of the type $\mathcal{A} = L^\infty(\mathbb{R},\mathfrak{B})$ with $\mathfrak{B}$ a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$. In this case, $\Sigma_\delta(L,\mathcal{A})$ corresponds to a switched system under arbitrary $\mathfrak{B}$-valued switching signals (for a general discussion on switched systems and their stability, see e.g. [113, 167] and references therein).

Motivated by formula (4.14) for $\widehat{\Xi}_{[\mathbf{n}],t}$, we define below a new measure of the asymptotic behavior of $\Sigma_\delta(L,\mathcal{A})$. For this, we introduce, for $\Lambda \in (\mathbb{R}_+^*)^N$ and $x \in \mathbb{R}_+$,

$$\mathcal{L}(\Lambda) = \{\Lambda\cdot\mathbf{n} \mid \mathbf{n}\in\mathbb{N}^N\} \qquad \text{and} \qquad \mathcal{L}_x(\Lambda) = \mathcal{L}(\Lambda)\cap[0,x). \tag{4.27}$$

Definition 4.28. We define

$$\mu(\Lambda,\mathfrak{B}) = \limsup_{\substack{x\to+\infty\\ x\in\mathcal{L}(\Lambda)}} \sup_{\substack{B^r\in\mathfrak{B}\\ \text{for } r\in\mathcal{L}_x(\Lambda)}} \left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}\right|^{\frac{1}{x}}.$$

Note that $\mu(\Lambda,\mathfrak{B})$ is independent of the choice of the norm $|\cdot|$ and $\mu(\Lambda,\mathfrak{B}) = \mu(\Lambda,\overline{\mathfrak{B}})$. The main result of this section is the following.

Theorem 4.29. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, $\mathcal{A} = L^\infty(\mathbb{R},\mathfrak{B})$, and $p\in[1,+\infty]$. Set $m_1 = \min_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j}$ and $m_2 = \max_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j}$ if $\mu(\Lambda,\mathfrak{B}) < 1$, and $m_1 = \max_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j}$ and $m_2 = \min_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j}$ if $\mu(\Lambda,\mathfrak{B}) \geq 1$. Then the following assertions hold:*

(a) *$\lambda_p(L,\mathcal{A}) \leq m_1 \log\mu(\Lambda,\mathfrak{B})$;*

(b) *if $L \in W_+(\Lambda)$, then $m_2\lambda_p(\Lambda,\mathcal{A}) \leq \lambda_p(L,\mathcal{A}) \leq m_1\lambda_p(\Lambda,\mathcal{A})$;*

(c) *$\lambda_p(\Lambda,\mathcal{A}) = \log\mu(\Lambda,\mathfrak{B})$.*

Proof. Notice that (b) follows from (a) and (c) by exchanging the role of L and Λ, since $\Lambda \in V_+(L)$ for every $L \in W_+(\Lambda)$.

Let us prove (a). Since $\min_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j} \leq \frac{\Lambda\cdot\mathbf{n}}{L\cdot\mathbf{n}} \leq \max_{j\in[\![1,N]\!]} \frac{\Lambda_j}{L_j}$ for every $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, it suffices to show that, for every $\varepsilon > 0$, there exists $C > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N\setminus\{0\}$, and $t \in \mathbb{R}$, we have

$$\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq C\left(\mu(\Lambda,\mathfrak{B}) + \varepsilon\right)^{\Lambda\cdot\mathbf{n}}. \tag{4.28}$$

By definition of $\mu(\Lambda,\mathfrak{B})$, there exists $X_0 \in \mathcal{L}(\Lambda)$ such that, for every $x \in \mathcal{L}(\Lambda)$ with $x \geq X_0$, we have

$$\sup_{\substack{B^r\in\mathfrak{B}\\ \text{for } r\in\mathcal{L}_x(\Lambda)}} \left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}\right| \leq \left(\mu(\Lambda,\mathfrak{B}) + \varepsilon\right)^x.$$

Since $\mathfrak{B}$ is bounded, the quantity

$$C' = \max_{x\in\mathcal{L}_{X_0}(\Lambda)} \sup_{\substack{B^r\in\mathfrak{B}\\ \text{for } r\in\mathcal{L}_x(\Lambda)}} \left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}\right|$$

is finite. Setting $C = \max\{1, C', C'(\mu(\Lambda,\mathfrak{B}) + \varepsilon)^{-X_0}\}$, we have, for every $x \in \mathcal{L}(\Lambda)$,

$$\sup_{\substack{B^r\in\mathfrak{B}\\ \text{for } r\in\mathcal{L}_x(\Lambda)}} \left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=r}} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}\right| \leq C\left(\mu(\Lambda,\mathfrak{B}) + \varepsilon\right)^x. \tag{4.29}$$

Define $\varphi_L : \mathcal{L}(\Lambda) \to \mathcal{L}(L)$ by $\varphi_L(\Lambda\cdot\mathbf{n}) = L\cdot\mathbf{n}$. This is a well-defined function since $L \in V_+(\Lambda)$. Let $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$. By Proposition 4.12,

$$\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t} = \sum_{\mathbf{n}'\in[\mathbf{n}]\cap\mathbb{N}^N} \sum_{v\in V_{\mathbf{n}'}} \prod_{k=1}^{|\mathbf{n}'|_1} A_{v_k}(t - L\cdot\mathbf{p}_v(k)). \tag{4.30}$$

For $r \in \mathcal{L}_{\Lambda\cdot\mathbf{n}}(\Lambda)$, we set $B^r = A(t - \varphi_L(r)) \in \mathfrak{B}$. Thus, for every $\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N$, $v \in V_{\mathbf{n}'}$, and $k \in [\![1, |\mathbf{n}'|_1]\!]$, we have, by definition of φ_L,

$$B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)} = A_{v_k}(t - \varphi_L(\Lambda\cdot\mathbf{p}_v(k))) = A_{v_k}(t - L\cdot\mathbf{p}_v(k)). \tag{4.31}$$

We thus obtain (4.28) by combining (4.29), (4.30) and (4.31).

In order to prove (c), we are left to show the inequality $\log\mu(\Lambda,\mathfrak{B}) \leq \lambda_p(\Lambda,\mathcal{A})$. Let $x \in \mathcal{L}(\Lambda)$ and $A^0 \in \mathfrak{B}$. For $r \in \mathcal{L}_x(\Lambda)$, let $B^r \in \mathfrak{B}$. We define

$$\zeta = \frac{1}{2} \min_{\substack{y_1,y_2\in\mathcal{L}_x(\Lambda)\\ y_1\neq y_2}} \left|y_1 - y_2\right| > 0.$$

Let $A = (A_1,\dots,A_N) \in \mathcal{A}$ be defined for $t \in \mathbb{R}$ by

$$A(t) = \begin{cases} B^{\Lambda\cdot\mathbf{m}}, & \text{if } \mathbf{m}\in\mathbb{N}^N \text{ is such that } \Lambda\cdot\mathbf{m} < x \text{ and } t \in (-\Lambda\cdot\mathbf{m} - \zeta, -\Lambda\cdot\mathbf{m} + \zeta),\\ A^0, & \text{otherwise.}\end{cases}$$

The function A is well-defined since the sets $(-\Lambda\cdot\mathbf{m}-\zeta,-\Lambda\cdot\mathbf{m}+\zeta)$ are disjoint for $\mathbf{m}\in\mathbb{N}^N$ with $\Lambda\cdot\mathbf{m}<x$. For every $\mathbf{n}\in\mathbb{N}^N$ with $\Lambda\cdot\mathbf{n}=x$, every $v\in V_{\mathbf{n}}$, $t\in(-\zeta,\zeta)$, and $k\in[\![1,|\mathbf{n}|_1]\!]$, we have

$$A_{v_k}(t-\Lambda\cdot\mathbf{p}_v(k))=B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)},$$

and then, for every $\mathbf{n}'\in\mathbb{N}^N$ with $\Lambda\cdot\mathbf{n}'=x$, we have

$$\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}}\sum_{v\in V_{\mathbf{n}}}\prod_{k=1}^{|\mathbf{n}|_1}B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}=\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}}\sum_{v\in V_{\mathbf{n}}}\prod_{k=1}^{|\mathbf{n}|_1}A_{v_k}(t-\Lambda\cdot\mathbf{p}_v(k))=\widehat{\Xi}_{[\mathbf{n}'],t}^{\Lambda,\Lambda,A}.$$

Hence, for every $\mathbf{n}'\in\mathbb{N}^N$ with $\Lambda\cdot\mathbf{n}'=x$, we have

$$\left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}=x}}\sum_{v\in V_{\mathbf{n}}}\prod_{k=1}^{|\mathbf{n}|_1}B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)}\right|^{\frac{1}{x}}\leq\sup_{A\in\mathcal{A}}\operatorname*{ess\,sup}_{t\in\mathbb{R}}\left|\widehat{\Xi}_{[\mathbf{n}'],t}^{\Lambda,\Lambda,A}\right|^{\frac{1}{\Lambda\cdot\mathbf{n}'}}.$$

Since this holds for every choice of $B^r\in\mathfrak{B}$, $r\in\mathcal{L}_x(\Lambda)$, and $x\in\mathcal{L}(\Lambda)$, we deduce from (4.26) that $\log\mu(\Lambda,\mathfrak{B})\leq\lambda_p(\Lambda,\mathcal{A})$. ■

Remark 4.30. Since $\mu(\Lambda,\mathfrak{B})=\mu(\Lambda,\overline{\mathfrak{B}})$, one has $\lambda_p(\Lambda,\mathcal{A})=\lambda_p(\Lambda,L^\infty(\mathbb{R},\overline{\mathfrak{B}}))$.

As regards exponential stability of $\Sigma_\delta(L,\mathcal{A})$, we deduce from the previous theorem and Remark 4.17 the following corollary.

Corollary 4.31. *Let $\Lambda\in(\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A}=L^\infty(\mathbb{R},\mathfrak{B})$. The following statements are equivalent:*

(a) *$\mu(\Lambda,\mathfrak{B})<1$;*

(b) *$\Sigma_\delta(\Lambda,\mathcal{A})$ is exponentially stable in X_p^δ for some $p\in[1,+\infty]$;*

(c) *$\Sigma_\delta(L,\mathcal{A})$ is exponentially stable in X_p^δ for every $L\in V_+(\Lambda)$ and $p\in[1,+\infty]$.*

Moreover, for every $p\in[1,+\infty]$,

$$\lambda_p(\Lambda,\mathcal{A})=\inf\{\nu\in\mathbb{R}\mid\mu(\Lambda,\mathfrak{B}_{-\nu})<1\},$$

where $\mathfrak{B}_{-\nu}=\{(e^{-\nu\Lambda_1}B_1,\dots,e^{-\nu\Lambda_N}B_N)\mid(B_1,\dots,B_N)\in\mathfrak{B}\}$.

Corollary 4.31 is reminiscent of the well-known characterization of stability in the autonomous case proved by Hale and Silkowski when Λ has rationally independent components (see Theorem 1.39) and in a more general setting by Michiels *et al.* in [132]. In such a characterization, $(1,\dots,1)$ is assumed to be in $V(\Lambda)$ and $\mu(\Lambda,\mathfrak{B})$ is replaced in the statement of Corollary 4.31 by

$$\rho_{\mathrm{HS}}(\Lambda,A)=\max_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)}\rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right),$$

where $\widetilde{V}(\Lambda)$ is the image of $V(\Lambda)$ by the canonical projection from $\mathbb{R}^N$ onto the torus $\mathbb{T}^N=(\mathbb{R}/2\pi\mathbb{Z})^N$. (Notice that $\widetilde{V}(\Lambda)$ is compact since the matrix B characterizing $V(\Lambda)$ in Proposition 4.9 has integer coefficients.)

We propose below a generalization of $\rho_{\mathrm{HS}}(\Lambda,A)$ to the non-autonomous case defined as follows.

Definition 4.32. For $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{L}(\Lambda)$ given by (4.27), we set

$$\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) = \limsup_{n\to+\infty} \sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \sup_{\substack{B^r\in\mathfrak{B} \\ \text{for } r\in\mathcal{L}_{n\Lambda_{\max}}(\Lambda)}} \left| \sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^{n} B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)} e^{i\theta_{v_k}} \right|^{\frac{1}{n}}.$$

Let us check in the next proposition that μ_{HS} actually extends ρ_{HS}.

Proposition 4.33. *Let $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $\mathfrak{B} = \{A\}$. Then one has $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) = \rho_{\mathrm{HS}}(\Lambda, A)$.*

Proof. One has

$$\begin{aligned}
\max_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \rho\left(\sum_{j=1}^{N} A_j e^{i\theta_j}\right) &= \max_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \lim_{n\to+\infty} \left| \left(\sum_{j=1}^{N} A_j e^{i\theta_j}\right)^n \right|^{\frac{1}{n}} \\
&= \lim_{n\to+\infty} \sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \left| \left(\sum_{j=1}^{N} A_j e^{i\theta_j}\right)^n \right|^{\frac{1}{n}} \\
&= \lim_{n\to+\infty} \sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \left| \sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^{n} A_{v_k} e^{i\theta_{v_k}} \right|^{\frac{1}{n}},
\end{aligned}$$

where the second equality is obtained as consequence of the uniformity of the Gelfand limit on bounded subsets of $\mathcal{M}_d(\mathbb{C})$ (see Lemma 2.32). ∎

In the sequel, we relate $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$ to a modified version of the expression (4.26) of $\lambda_p(L, \mathcal{A})$.

Definition 4.34. For $L \in V_+(\Lambda)$ and $\mathcal{A}$ a set of functions $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, we define

$$\lambda_{\mathrm{HS}}(L, \mathcal{A}) = \limsup_{\substack{|\mathbf{n}|_1\to+\infty \\ \mathbf{n}\in\mathbb{N}^N}} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{\log\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{|\mathbf{n}|_1}.$$

Remark 4.35. Since $L_{\min}|\mathbf{n}|_1 \le L\cdot\mathbf{n} \le L_{\max}|\mathbf{n}|_1$ for every $L \in V_+(\Lambda)$ and $\mathbf{n} \in \mathbb{N}^N$, it follows immediately from (4.26) that, for every $p \in [1, +\infty]$,

$$\begin{aligned}
L_{\min}\lambda_p(L,\mathcal{A}) \le \lambda_{\mathrm{HS}}(L,\mathcal{A}) &\le L_{\max}\lambda_p(L,\mathcal{A}), && \text{if } \lambda_p(L,\mathcal{A}) \ge 0, \\
L_{\max}\lambda_p(L,\mathcal{A}) \le \lambda_{\mathrm{HS}}(L,\mathcal{A}) &\le L_{\min}\lambda_p(L,\mathcal{A}), && \text{if } \lambda_p(L,\mathcal{A}) < 0.
\end{aligned}$$

In particular, the signs of $\lambda_{\mathrm{HS}}(L,\mathcal{A})$ and $\lambda_p(L,\mathcal{A})$ being equal, they both characterize the exponential stability of $\Sigma_\delta(L,\mathcal{A})$.

Theorem 4.36. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. Set $m = \inf\left\{1, \frac{|z_+|_1}{|z_-|_1} \;\middle|\; z \in Z(\Lambda)\setminus\{0\}\right\}$ if $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) < 1$ and $m = \sup\left\{1, \frac{|z_+|_1}{|z_-|_1} \;\middle|\; z \in Z(\Lambda)\setminus\{0\}\right\}$ if $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) \ge 1$. Then the following assertions hold:*

(a) *for every $L \in V_+(\Lambda)$, $\lambda_{\mathrm{HS}}(L,\mathcal{A}) \le m \log \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$;*

(b) *if $(1,\dots,1) \in V(\Lambda)$ and $L \in W_+(\Lambda)$, one has $\lambda_{\mathrm{HS}}(L,\mathcal{A}) = \log \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$.*

Proof. We start by proving (a). It is enough to show that, for every $\varepsilon > 0$ small enough, there exists $C > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$, we have

$$\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq C(1+|\mathbf{n}|_1)\left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^{m|\mathbf{n}|_1}.$$

Let $L \in V_+(\Lambda)$ and $\varepsilon > 0$ be such that $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon < 1$ if $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B}) < 1$. We can proceed as in the proof of Theorem 4.29 to obtain a finite constant $C_0 > 0$ such that, for every $n \in \mathbb{N}^*$,

$$\sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \sup_{\substack{B^r\in\mathfrak{B}\\ \text{for } r\in\mathcal{L}_{n\Lambda_{\max}}(\Lambda)}} \left|\sum_{v\in[\![1,N]\!]^n}\prod_{k=1}^n B^{\Lambda\cdot\mathbf{p}_v(k)}_{v_k} e^{i\theta_{v_k}}\right| \leq C_0\left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n. \tag{4.32}$$

Let $A \in \mathcal{A}$, $t \in \mathbb{R}$, and φ_L be as in the proof of Theorem 4.29. For $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$, we set $B^r = A(t-\varphi_L(r))$, and similarly to the proof of Theorem 4.29, (4.31) holds for every $v \in [\![1,N]\!]^n$ and $k \in [\![1,n]\!]$. Thus (4.32) implies that, for every $n \in \mathbb{N}^*$ and $\theta \in \widetilde{V}(\Lambda)$,

$$\left|\sum_{v\in[\![1,N]\!]^n}\prod_{k=1}^n A_{v_k}(t-L\cdot\mathbf{p}_v(k))e^{i\theta_{v_k}}\right| \leq C_0\left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n.$$

Since

$$\begin{aligned}\sum_{v\in[\![1,N]\!]^n}\prod_{k=1}^n A_{v_k}(t-L\cdot\mathbf{p}_v(k))e^{i\theta_{v_k}} &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\sum_{v\in V_{\mathbf{n}}}\prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}(t-L\cdot\mathbf{p}_v(k))e^{i\theta_{v_k}}\\ &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta}\sum_{v\in V_{\mathbf{n}}}\prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}(t-L\cdot\mathbf{p}_v(k)) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta}\Xi^{L,A}_{\mathbf{n},t},\end{aligned}$$

we obtain that, for every $n \in \mathbb{N}^*$ and $\theta \in \widetilde{V}(\Lambda)$,

$$\left|\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta}\Xi^{L,A}_{\mathbf{n},t}\right| \leq C_0\left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n. \tag{4.33}$$

Following Proposition 4.9, fix $h \in [\![1,N]\!]$ and $B \in \mathcal{M}_{N,h}(\mathbb{Z})$ with $\mathrm{rk}(B) = h$ such that $\Lambda = B\ell_0$ for $\ell_0 \in (\mathbb{R}_+^*)^h$ with rationally independent components. Let $M \in \mathrm{GL}_h(\mathbb{R})$ be such that $\ell_0 = Me_1$, where e_1 is the first vector of the canonical basis of $\mathbb{R}^h$, in such a way that $\Lambda = BMe_1$. For $n \in \mathbb{N}$, we define the function $f_n : \mathbb{R}^h \to \mathcal{M}_d(\mathbb{C})$ by

$$f_n(\nu) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot BM\nu}\Xi^{L,A}_{\mathbf{n},t}.$$

We claim that, for every $\mathbf{n}_0 \in \mathbb{N}^N$,

$$\lim_{R\to+\infty}\frac{1}{(2R)^h}\int_{[-R,R]^h} f_n(\nu)e^{-i\mathbf{n}_0\cdot BM\nu}\,d\nu = \sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\Xi^{L,A}_{\mathbf{n},t}. \tag{4.34}$$

Indeed, we have

$$\frac{1}{(2R)^h}\int_{[-R,R]^h} f_n(\nu)e^{-i\mathbf{n}_0\cdot BM\nu}\,d\nu=\sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\Xi_{\mathbf{n},t}^{L,A}\frac{1}{(2R)^h}\int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu}\,d\nu.$$

If $\mathbf{n}\in\mathbb{N}^N$ is such that $\Lambda\cdot\mathbf{n}=\Lambda\cdot\mathbf{n}_0$, then $\Lambda\cdot(\mathbf{n}-\mathbf{n}_0)=0$, and therefore $\mathbf{n}-\mathbf{n}_0\in Z(\Lambda)\subset V(\Lambda)^\perp=(\operatorname{Ran}B)^\perp$. One gets $(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu=0$ for every $\nu\in\mathbb{R}^h$, implying that

$$\frac{1}{(2R)^h}\int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu}\,d\nu=1.$$

If now $\Lambda\cdot\mathbf{n}\neq\Lambda\cdot\mathbf{n}_0$, set $\xi=\Lambda\cdot(\mathbf{n}-\mathbf{n}_0)$, which is nonzero. Then

$$\left|\frac{1}{(2R)^h}\int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu}\,d\nu\right|\leq\frac{1}{2R}\left|\int_{-R}^{R}e^{i\xi\nu_1}\,d\nu_1\right|=\left|\frac{\sin(\xi R)}{\xi R}\right|\xrightarrow[R\to+\infty]{}0,$$

which gives (4.34).

We can now combine (4.33) and (4.34) to obtain that, for every $n\in\mathbb{N}^*$ and $\mathbf{n}_0\in\mathbb{N}^N\setminus\{0\}$,

$$\left|\sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\Xi_{\mathbf{n},t}^{L,A}\right|\leq C_0(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon)^n.$$

Set $m_0=\sup\left\{1,\frac{|z_+|_1}{|z_-|_1}\,\middle|\,z\in Z(\Lambda)\setminus\{0\}\right\}$ and notice that, since $Z(\Lambda)=-Z(\Lambda)$, one has $\frac{1}{m_0}=\inf\left\{1,\frac{|z_+|_1}{|z_-|_1}\,\middle|\,z\in Z(\Lambda)\setminus\{0\}\right\}$. We claim that, if $\mathbf{n},\mathbf{n}_0\in\mathbb{N}^N$ and $\Lambda\cdot\mathbf{n}=\Lambda\cdot\mathbf{n}_0$, then $\frac{1}{m_0}|\mathbf{n}_0|_1\leq|\mathbf{n}|_1\leq m_0|\mathbf{n}_0|_1$. Indeed, let $z=\mathbf{n}-\mathbf{n}_0\in Z(\Lambda)$ and $\mathbf{n}_1=\mathbf{n}_0-z_-\in\mathbb{N}^N$. Then one has

$$\frac{|\mathbf{n}|_1}{|\mathbf{n}_0|_1}=\frac{|z_+|_1+|\mathbf{n}_1|_1}{|z_-|_1+|\mathbf{n}_1|_1}\in\left[\frac{1}{m_0},\,m_0\right].$$

Hence, for every $\mathbf{n}_0\in\mathbb{N}^N\setminus\{0\}$,

$$\widehat{\Xi}_{[\mathbf{n}_0],t}^{L,\Lambda,A}=\sum_{n=0}^{+\infty}\sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\Xi_{\mathbf{n},t}^{L,A}=\sum_{n\in\left[\!\left[\frac{|\mathbf{n}_0|_1}{m_0},\,m_0|\mathbf{n}_0|_1\right]\!\right]}\sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ |\mathbf{n}|_1=n}}\Xi_{\mathbf{n},t}^{L,A},$$

and we conclude that

$$\left|\widehat{\Xi}_{[\mathbf{n}_0],t}^{L,\Lambda,A}\right|\leq\sum_{n\in\left[\!\left[\frac{|\mathbf{n}_0|_1}{m_0},\,m_0|\mathbf{n}_0|_1\right]\!\right]}C_0(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon)^n\leq C(1+|\mathbf{n}_0|_1)(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon)^{m|\mathbf{n}_0|_1},$$

for some $C>0$. This concludes the proof of (a).

Suppose now that $(1,\dots,1)\in V(\Lambda)$. Then $|z_+|_1=|z_-|_1$ for every $z\in Z(\Lambda)$, and hence (a) yields $\lambda_{\mathrm{HS}}(L,\mathcal{A})\leq\log\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})$ for every $L\in V_+(\Lambda)$. We claim that it is enough to prove (b) only for $L=\Lambda$. Indeed, assume that $\lambda_{\mathrm{HS}}(\Lambda,\mathcal{A})=\log\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})$. In particular,

$$\lambda_{\mathrm{HS}}(L,\mathcal{A})\leq\lambda_{\mathrm{HS}}(\Lambda,\mathcal{A})\tag{4.35}$$

for every $L\in V_+(\Lambda)$. Since $\Lambda\in V_+(L)$ if $L\in W_+(\Lambda)$, by exchanging the role of L and Λ in (4.35), we deduce that $\lambda_{\mathrm{HS}}(L,\mathcal{A})=\lambda_{\mathrm{HS}}(\Lambda,\mathcal{A})$ for every $L\in W_+(\Lambda)$, and hence (b).

Let $n \in \mathbb{N}^*$ and $B^r \in \mathfrak{B}$ for $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$. As in the argument for (c) in Theorem 4.29, there exist $\zeta > 0$ and a function $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$ such that, for every $v \in [\![1, N]\!]^n$, $t \in (-\zeta, \zeta)$, and $k \in [\![1, n]\!]$, we have

$$A_{v_k}(t - \Lambda \cdot \mathbf{p}_v(k)) = B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \quad \text{and} \quad \sum_{v \in [\![1,N]\!]^n} \prod_{k=1}^n B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} e^{i\theta_{v_k}} = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ |\mathbf{n}|_1 = n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{\Lambda,A}.$$

Denote $\mathcal{Z}_+ = \{[\mathbf{n}] \in \mathcal{Z} \mid [\mathbf{n}] \cap \mathbb{N}^N \neq \emptyset\}$. Since $(1, \dots, 1) \in V(\Lambda)$, one deduces that, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\mathbf{n} \approx \mathbf{n}'$, then $e^{i\mathbf{n}\cdot\theta} = e^{i\mathbf{n}'\cdot\theta}$ for every $\theta \in \widetilde{V}(\Lambda)$ and $|\mathbf{n}|_1 = |\mathbf{n}'|_1$. We set $|[\mathbf{n}]|_1 = |\mathbf{n}|_1$ for every $\mathbf{n} \in \mathbb{N}^N$. Then

$$\sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ |\mathbf{n}|_1 = n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z}_+ \\ |[\mathbf{n}]|_1 = n}} \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} e^{i\mathbf{n}'\cdot\theta} \Xi_{\mathbf{n}',t}^{\Lambda,A} = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z}_+ \\ |[\mathbf{n}]|_1 = n}} e^{i\mathbf{n}\cdot\theta} \widehat{\Xi}_{[\mathbf{n}],t}^{\Lambda,\Lambda,A}.$$

We clearly have $\#\{[\mathbf{n}] \in \mathcal{Z}_+ \mid |[\mathbf{n}]|_1 = n\} \leq \#\{\mathbf{n} \in \mathbb{N}^N \mid |\mathbf{n}|_1 = n\} = \binom{n+N-1}{N-1} \leq (n+1)^{N-1}$, and we get that, for every $\theta \in \widetilde{V}(\Lambda)$ and $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 = n$,

$$\left| \sum_{v \in [\![1,N]\!]^n} \prod_{k=1}^n B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} e^{i\theta_{v_k}} \right|^{\frac{1}{n}} \leq (n+1)^{\frac{N-1}{n}} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \left| \widehat{\Xi}_{[\mathbf{n}],t}^{\Lambda,\Lambda,A} \right|^{\frac{1}{n}}.$$

Since the above inequality holds for every choice of $B^r \in \mathfrak{B}$, $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$, $n \in \mathbb{N}^*$, we deduce that $\log \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) \leq \lambda_{\mathrm{HS}}(\Lambda, \mathcal{A})$. This concludes the proof of Theorem 4.36. ■

The next corollary, which follows directly from the above theorem and Remarks 4.17 and 4.35, generalizes the stability criterion from Theorem 1.39 and its counterpart in [132] to the nonautonomous case (see Proposition 4.33).

Corollary 4.37. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. Consider the following statements:*

(a) $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) < 1$;

(b) *$\Sigma_\delta(\Lambda, \mathcal{A})$ is exponentially stable in X_p^δ for some $p \in [1, +\infty]$;*

(c) *$\Sigma_\delta(L, \mathcal{A})$ is exponentially stable in X_p^δ for every $L \in V_+(\Lambda)$ and $p \in [1, +\infty]$.*

Then (a) $\implies$ (c) $\implies$ (b). If moreover $(1, \dots, 1) \in V(\Lambda)$, we also have (b) $\implies$ (a) and, for every $p \in [1, +\infty]$,

$$\lambda_p(\Lambda, \mathcal{A}) = \inf\{\nu \in \mathbb{R} \mid \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}_{-\nu}) < 1\},$$

where $\mathfrak{B}_{-\nu} = \{(e^{-\nu\Lambda_1} B_1, \dots, e^{-\nu\Lambda_N} B_N) \mid (B_1, \dots, B_N) \in \mathfrak{B}\}$.

4.3 Transport system

For $L = (L_1, \dots, L_N) \in (\mathbb{R}_+^*)^N$ and $M = (m_{ij})_{i,j \in [\![1,N]\!]} : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, we consider the system of transport equations

$$\Sigma_\tau(L, M) : \begin{cases} \dfrac{\partial u_i}{\partial t}(t, x) + \dfrac{\partial u_i}{\partial x}(t, x) = 0, & i \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_i], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, L_j), & i \in [\![1, N]\!],\ t \in [0, +\infty), \end{cases} \tag{4.36}$$

where, for $i \in \llbracket 1, N \rrbracket$, $u_i(\cdot,\cdot)$ takes values in $\mathbb{C}$.

The time-varying matrix M represents transmission conditions and in particular it may encode an underlying network for (4.36), where the graph structure is determined by the non-zero coefficients of M. When no regularity assumptions are made on the function M, we may not have solutions for (4.36) in the classical sense in $C^1(\mathbb{R}_+ \times [0, L_i])$ nor in $C^0(\mathbb{R}_+, W^{1,p}([0, L_i], \mathbb{C})) \cap C^1(\mathbb{R}_+, L^p([0, L_i], \mathbb{C}))$. We thus consider the following weaker definition of solution.

Definition 4.38. Let $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$ and $u_{i,0} : [0, L_i] \to \mathbb{C}$ for $i \in \llbracket 1, N \rrbracket$. We say that $(u_i)_{i\in\llbracket 1,N\rrbracket}$ is a *solution* of $\Sigma_\tau(L, M)$ with initial condition $(u_{i,0})_{i\in\llbracket 1,N\rrbracket}$ if $u_i : \mathbb{R}_+ \times [0, L_i] \to \mathbb{C}$, $i \in \llbracket 1, N \rrbracket$, satisfy the second equation of (4.36), and, for every $i \in \llbracket 1, N \rrbracket$, $t \geq 0$, $x \in [0, L_i]$, $s \in [-\min(x, t), L_i - x]$, one has $u_i(t + s, x + s) = u_i(t, x)$ and $u_i(0, x) = u_{i,0}(x)$.

4.3.1 Equivalent difference equation

For $i \in \llbracket 1, N \rrbracket$ and $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, define the orthogonal projection $P_i = e_i e_i^{\mathrm{T}}$ and set $A_i(\cdot) = M(\cdot)P_i$. Consider the system of difference equations

$$v(t) = \sum_{j=1}^{N} A_j(t) v(t - L_j). \tag{4.37}$$

This system is equivalent to (4.36) in the following sense.

Proposition 4.39. *Suppose that $u = (u_i)_{i\in\llbracket 1,N\rrbracket}$ is a solution of (4.36) with initial condition $(u_{i,0})_{i\in\llbracket 1,N\rrbracket}$ and let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be given for $i \in \llbracket 1, N \rrbracket$ by*

$$v_i(t) = \begin{cases} 0, & \text{if } t \in [-L_{\max}, -L_i), \\ u_{i,0}(-t), & \text{if } t \in [-L_i, 0), \\ u_i(t, 0), & \text{if } t \geq 0. \end{cases} \tag{4.38}$$

Then v is a solution of (4.37).

Conversely, suppose that $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ is a solution of (4.37) and let $u = (u_i)_{i\in\llbracket 1,N\rrbracket}$ be given for $i \in \llbracket 1, N \rrbracket$, $t \geq 0$ and $x \in [0, L_i]$ by $u_i(t, x) = v_i(t - x)$. Then $(u_i)_{i\in\llbracket 1,N\rrbracket}$ is a solution of (4.36).

Proof. Let $(u_i)_{i\in\llbracket 1,N\rrbracket}$ be a solution of (4.36) with initial condition $(u_{i,0})_{i\in\llbracket 1,N\rrbracket}$ and let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be given by (4.38). Then, for $t \geq 0$,

$$v_i(t) = u_i(t, 0) = \sum_{j=1}^{N} m_{ij}(t) u_j(t, L_j),$$

and, by Definition 4.38, $u_j(t, L_j) = v_j(t - L_j)$ since $u_j(t, L_j) = u_j(t - L_j, 0)$ if $t \geq L_j$ and $u_j(t, L_j) = u_{j,0}(L_j - t)$ if $0 \leq t < L_j$. Hence $v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j)$ and thus $v(t) = \sum_{j=1}^{N} A_j(t) v(t - L_j)$.

Conversely, suppose that $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ is a solution of (4.37) with initial condition v_0 and let $(u_i)_{i\in\llbracket 1,N\rrbracket}$ be given for $i \in \llbracket 1, N \rrbracket$, $t \geq 0$ and $x \in [0, L_i]$ by $u_i(t, x) = v_i(t - x)$. It is then clear that $u_i(t + s, x + s) = u_i(t, x)$ for $s \in [-\min(x, t), L_i - x]$, and, since $v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j)$,

$$u_i(t, 0) = v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j) = \sum_{j=1}^{N} m_{ij}(t) u_j(t, L_j),$$

and so $(u_i)_{i\in\llbracket 1,N\rrbracket}$ is a solution of (4.36). ■

The following result follows immediately from Proposition 4.2.

Proposition 4.40. *Let $u_{i,0} : [0, L_i] \to \mathbb{C}$ for $i \in [\![1, N]\!]$ and $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$. Then $\Sigma_\tau(L, M)$ admits a unique solution $(u_i)_{i\in[\![1,N]\!]}$, $u_i : \mathbb{R}_+ \times [0, L_i] \to \mathbb{C}$ for $i \in [\![1, N]\!]$, with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$.*

4.3.2 Invariant subspaces

For $p \in [1, +\infty]$, consider (4.36) in the Banach space

$$\mathsf{X}_p^\tau = \prod_{i=1}^{N} L^p([0, L_i], \mathbb{C})$$

endowed with the norm

$$\|u\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^{N} \|u_i\|_{L^p([0,L_i],\mathbb{C})}^p \right)^{1/p}, & \text{if } p \in [1, +\infty), \\ \displaystyle\max_{i\in[\![1,N]\!]} \|u_i\|_{L^\infty([0,L_i],\mathbb{C})}, & \text{if } p = +\infty. \end{cases}$$

It follows from Proposition 4.39 and Remark 4.4 that, if $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$ and $u_0 \in \mathsf{X}_p^\tau$, then the solution $t \mapsto u(t)$ of $\Sigma_\tau(L, M)$ with initial condition u_0 takes values in X_p^τ for every $t \geq 0$.

In Section 4.4, we study wave propagation on networks using transport equations via the d'Alembert decomposition. For that purpose, we need to study transport equations in the range of the d'Alembert decomposition operator, which happens to take the following form (see Proposition 4.52). For $r \in \mathbb{N}$ and $R = (\rho_{ij})_{i\in[\![1,r]\!], j\in[\![1,N]\!]} \in \mathcal{M}_{r,N}(\mathbb{C})$, let

$$\mathsf{Y}_p(R) = \left\{ u = (u_1, \dots, u_N) \in \mathsf{X}_p^\tau \;\middle|\; \forall i \in [\![1, r]\!],\ \sum_{j=1}^{N} \rho_{ij} \int_0^{L_j} u_j(x) dx = 0 \right\}.$$

This is a closed subspace of X_p^τ, which is thus itself a Banach space.

Remark 4.41. Let $r \in \mathbb{N}$, $R \in \mathcal{M}_{r,N}(\mathbb{C})$, and $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Note that, if $1 \leq p \leq q \leq +\infty$, $\mathsf{Y}_q(R)$ is a dense subset of $\mathsf{Y}_p(R)$ since X_q^τ is a dense subset of X_p^τ. As a consequence, by a density argument, Propositions 4.14 and 4.39, one obtains that, if $\mathsf{Y}_p(R)$ is invariant under the flow of $\Sigma_\tau(L, M)$ for some $p \in [1, +\infty]$, then $\mathsf{Y}_q(R)$ is invariant for every $q \in [1, +\infty]$.

The following proposition provides a necessary and sufficient condition for $\mathsf{Y}_p(R)$ to be invariant under the flow of (4.36).

Proposition 4.42. *Let $r \in \mathbb{N}$, $R \in \mathcal{M}_{r,N}(\mathbb{C})$, $(u_{i,0})_{i\in[\![1,N]\!]} \in \mathsf{Y}_p(R)$, and $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Then the solution $u = (u_i)_{i\in[\![1,N]\!]}$ of $\Sigma_\tau(L, M)$ with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$ belongs to $\mathsf{Y}_p(R)$ for every $t \geq 0$ if and only if*

$$R(M(t) - \mathrm{Id}_N) w(t) = 0$$

for almost every $t \geq 0$, where $w = (w_i)_{i\in[\![1,N]\!]}$ and $w_i(t) = u_i(t, L_i)$.

Proof. Let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be the solution of (4.37) corresponding to u, given by (4.38), and let $w = (w_i)_{i\in[\![1,N]\!]}$ be defined by $w_i(t) = v_i(t - L_i) = u_i(t, L_i)$. Let $\lambda = (\lambda_i)_{i\in[\![1,r]\!]}$ be given for $i \in [\![1, r]\!]$ by $\lambda_i(t) = \sum_{j=1}^{N} \rho_{ij} \int_0^{L_j} u_j(t, x) dx$. Since $\lambda_i(0) = 0$, we have

$$\lambda_i(t) = \sum_{j=1}^{N} \rho_{ij} \left[\int_0^{L_j} u_j(t, x) dx - \int_0^{L_j} u_{j,0}(x) dx \right] = \sum_{j=1}^{N} \rho_{ij} \left[\int_0^{L_j} v_j(t - x) dx - \int_0^{L_j} v_j(-x) dx \right]$$

$$= \sum_{j=1}^{N} \rho_{ij} \left[\int_{t-L_j}^{t} v_j(s) ds - \int_{0}^{L_j} v_j(s - L_j) ds \right] = \sum_{j=1}^{N} \rho_{ij} \int_0^t \left(v_j(s) - v_j(s - L_j) \right) ds$$
$$= \sum_{j=1}^{N} \rho_{ij} \int_0^t \left(\sum_{k=1}^{N} m_{jk}(s) v_k(s - L_k) - v_j(s - L_j) \right) ds = \sum_{j=1}^{N} \rho_{ij} \int_0^t \sum_{k=1}^{N} \left(m_{jk}(s) - \delta_{jk} \right) v_k(s - L_k) ds,$$

so that $\lambda(t) = \int_0^t R(M(s) - \mathrm{Id}_N) w(s) ds$. The conclusion follows immediately. ■

Definition 4.43. Let $L \in (\mathbb{R}_+^*)^N$ and $\mathcal{M}$ be a subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. We denote by $\mathrm{Inv}(\mathcal{M})$ the set

$$\mathrm{Inv}(\mathcal{M}) = \{R \in \mathcal{M}_{r,N}(\mathbb{C}) \mid r \in \mathbb{N},\ \mathsf{Y}_p(R) \text{ is invariant under the flow of } \Sigma_\tau(L, M),\ \forall M \in \mathcal{M},\ \forall p \in [1, +\infty]\}.$$

4.3.3 Stability of solutions on invariant subspaces

We next provide a definition for exponential stability of (4.36).

Definition 4.44. Let $p \in [1, +\infty]$, $L \in (\mathbb{R}_+^*)^N$, $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, and $R \in \mathrm{Inv}(\mathcal{M})$. Let $\Sigma_\tau(L, \mathcal{M})$ denote the family of systems $\Sigma_\tau(L, M)$ for $M \in \mathcal{M}$. We say that $\Sigma_\tau(L, \mathcal{M})$ is of *exponential type* γ in $\mathsf{Y}_p(R)$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $M \in \mathcal{M}$ and $u_0 \in \mathsf{Y}_p(R)$, the corresponding solution u of $\Sigma_\tau(L, M)$ satisfies, for every $t \geq 0$,

$$\|u(t)\|_p \leq K e^{(\gamma+\varepsilon)t} \|u_0\|_p.$$

We say that $\Sigma_\tau(L, \mathcal{M})$ is *exponentially stable* in $\mathsf{Y}_p(R)$ if it is of negative exponential type.

The next corollaries translate Propositions 4.20 and 4.21 into the framework of transport equations.

Corollary 4.45. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, and $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Suppose that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}_+^*$ such that, for every $M \in \mathcal{M}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, (4.19) holds with $A = (A_1, \ldots, A_N)$ given by $A_i = M P_i$. Then there exists a constant $C > 0$ such that, for every $M \in \mathcal{M}$, $p \in [1, +\infty]$, and $u_0 \in \mathsf{X}_p^\tau$, the corresponding solution u of $\Sigma_\tau(L, M)$ satisfies*

$$\|u(t)\|_p \leq C(t+1)^{N-1} \max_{s \in [t - L_{\max}, t]} f(s) \|u_0\|_p, \qquad \forall t \in \mathbb{R}_+.$$

Proof. Let $C > 0$ be as in the Proposition 4.20. Let $M \in \mathcal{M}$, $p \in [1, +\infty]$, $u_0 \in \mathsf{X}_p^\tau$, and u be the solution of $\Sigma_\tau(L, M)$ with initial condition u_0. Let v be the corresponding solution of (4.37), given by (4.38), with initial condition v_0. Notice that $\|u_0\|_p = \|v_0\|_p$ and, for every $t \geq 0$, $\|u(t)\|_p \leq \|v_t\|_p$. By Proposition 4.20, we have, for every $t \geq 0$,

$$\|u(t)\|_p \leq \|v_t\|_p \leq C(t+1)^{N-1} \max_{s \in [t - L_{\max}, t]} f(s) \|v_0\|_p = C(t+1)^{N-1} \max_{s \in [t - L_{\max}, t]} f(s) \|u_0\|_p,$$

which is the desired result. ■

Corollary 4.46. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in W_+(\Lambda)$, $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Suppose that there exist $M \in \mathcal{M}$, $\mathbf{n}_0 \in \mathbb{N}^N$, and a set of positive measure $S \subset (L \cdot \mathbf{n}_0 - L_{\max}, L \cdot \mathbf{n}_0)$ such that, for every $t \in S$, (4.21) is satisfied* *with $A = (A_1, \dots, A_N)$ given by $A_i = MP_i$. Then there exist a constant $C > 0$ independent of f, an initial condition $u_0 \in \mathsf{X}_\infty^\tau$, and $t > 0$ such that, for every $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$, the solution u of $\Sigma_\tau(L, M)$ with initial condition u_0 satisfies $u(s) \in \mathsf{Y}_p(R)$ for every $s \geq 0$ and*

$$\|u(t)\|_p > C \min_{s \in [t - L_{\max}, t]} f(s) \|u_0\|_p.$$

Proof. As in Proposition 4.21, since $L \in W_+(\Lambda)$, we can assume for the rest of the argument that $\Lambda = L$.

Let $C > 0$ be as in Proposition 4.21. We construct an initial condition $v_0 \in \mathsf{X}_p^\delta$ as follows: choose t_0 and j_0 as in Proposition 4.21 and verifying in addition $t_0 \neq L \cdot \mathbf{n}_0 - L_{j_0}$. Then pick $\delta > 0$ as in Proposition 4.21 and satisfying in addition $\delta < \left|t_0 - L \cdot \mathbf{n}_0 + L_{j_0}\right|$ and $\delta < L_{\min}/2$. Next, take $\mu \in L^\infty(\mathbb{R}, \mathbb{R})$ as in Proposition 4.21 and satisfying in addition $\int_{-\delta}^{\delta} \mu(s) ds = 0$. Finally, consider the initial condition $v_0(s) = \mu(s - t_0 + L \cdot \mathbf{n}_0) e_{j_0}$. As in (4.24), we still obtain that the solution v of (4.37) with initial condition v_0 satisfies, for $p \in [1, +\infty]$,

$$\left\|v_{t_0+\delta}\right\|_p \geq \left\|v_{t_0}\right\|_{L^p([-\delta,\delta],\mathbb{C}^N)} > C \min_{s \in [t_0 + \delta - L_{\max}, t_0 + \delta]} f(s) \|v_0\|_p. \tag{4.39}$$

Let u be the solution of (4.36) corresponding to v, in the sense of Proposition 4.39, and denote by $u_0 = (u_{i,0})_{i \in [\![1,N]\!]}$ its initial condition. Since $u_{i,0}(x) = v_i(-x)$, we have $u_0 \in \prod_{i=1}^N L^\infty([0, L_i], \mathbb{C})$. Furthermore, $u_{i,0} = 0$ for $i \neq j_0$ and $u_{j_0,0}(x) = v_{j_0}(-x) = \mu(L \cdot \mathbf{n}_0 - t_0 - x)$. By definition of δ, we must have either $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \subset [0, L_{j_0}]$ or $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \cap [0, L_{j_0}] = \emptyset$, but the latter case is impossible since we would then have $u_{j_0,0} = 0$, and thus $v(s) = 0$ for every $s \geq -L_{\max}$, which contradicts (4.39). Hence $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \subset [0, L_{j_0}]$ and

$$\int_0^{L_{j_0}} u_{j_0,0}(x) dx = \int_{-\delta}^{\delta} \mu(x) dx = 0.$$

We thus have clearly $u_0 \in \mathsf{Y}_\infty(R)$, and in particular $u(s) \in \mathsf{Y}_p(R)$ for every $s \geq 0$ and $p \in [1, +\infty]$. Furthermore, $\|v_0\|_p = \|u_0\|_p$ and, for $p \in [1, +\infty)$,

$$\begin{aligned}
\left\|v_{t_0}\right\|_{L^p([-\delta,\delta],\mathbb{C}^N)}^p &= \int_{-\delta}^{\delta} |v(t_0 + s)|_p^p \, ds = \int_{-\delta}^{\delta} \sum_{i=1}^N |u_i(t_0 + s, 0)|^p \, ds = \int_0^{2\delta} \sum_{i=1}^N |u_i(t_0 + \delta, s)|^p \, ds \\
&\leq \sum_{i=1}^N \int_0^{L_i} |u_i(t_0 + \delta, s)|^p \, ds = \|u(t_0 + \delta)\|_p^p,
\end{aligned}$$

with a similar estimate for $p = +\infty$. Hence, it follows from (4.39) that, for every $p \in [1, +\infty]$,

$$\|u(t)\|_p > C \min_{s \in [t - L_{\max}, t]} f(s) \|u_0\|_p$$

with $t = t_0 + \delta$. ∎

As a consequence of the previous analysis, we have the following result.

Theorem 4.47. *Let $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, $\Lambda \in (\mathbb{R}_+^*)^N$, and $\mathcal{A} = \{A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i, M \in \mathcal{M}\}$. For every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ then, for every $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$, $\Sigma_\tau(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$. Conversely, for every $L \in W_+(\Lambda)$, if there exist $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$ such that $\Sigma_\tau(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$, then $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ.*

It follows from Theorem 4.47 that the exponential type γ for $\Sigma_\tau(L,\mathcal{M})$ in $\Upsilon_p(R)$ is independent of $p \in [1,+\infty]$ and $R \in \mathrm{Inv}(\mathcal{M})$. When $\mathcal{M}$ is shift-invariant, thanks to Theorem 4.26, one can replace (Θ,Λ)-exponential type by $(\widehat{\Xi},\Lambda)$-exponential type for $\Sigma_\delta(L,\mathcal{A})$ in Theorem 4.47.

Assume now that $\mathcal{M} = L^\infty(\mathbb{R},\mathfrak{B})$, where $\mathfrak{B}$ is a bounded subset of $\mathcal{M}_N(\mathbb{C})$. Let $\mathcal{A} = \{A = (A_1,\dots,A_N) : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i,\ M \in \mathcal{M}\}$, i.e., $\mathcal{A} = L^\infty(\mathbb{R},\mathfrak{A})$ where $\mathfrak{A} = \{A = (A_1,\dots,A_N) \in \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i,\ M \in \mathfrak{B}\}$. We can thus transpose the results from Section 4.2.3.3, and in particular Corollary 4.31, to the transport framework.

Corollary 4.48. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_N(\mathbb{C})$, $\mathcal{M} = L^\infty(\mathbb{R},\mathfrak{B})$. The following statements are equivalent.*

(a) *$\Sigma_\tau(\Lambda,\mathcal{M})$ is exponentially stable in $\Upsilon_p(R)$ for some $p \in [1,+\infty]$ and $R \in \mathrm{Inv}(\mathcal{M})$.*

(b) *$\Sigma_\tau(L,\mathcal{M})$ is exponentially stable in $\Upsilon_p(R)$ for every $L \in V_+(\Lambda)$, $p \in [1,+\infty]$, and $R \in \mathrm{Inv}(\mathcal{M})$.*

Remark 4.49. In accordance with Remark 4.30, the exponential stability of $\Sigma_\tau(\Lambda,\mathcal{M})$ is equivalent to that of $\Sigma_\tau(\Lambda,L^\infty(\mathbb{R},\overline{\mathfrak{B}}))$.

4.4 Wave propagation on networks

We consider here the problem of wave propagation on a finite network of elastic strings. The notations we use here come from [63].

A *graph* $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a set, whose elements are called *vertices*, and

$$\mathcal{E} \subset \{\{q,p\} \mid q,p \in \mathcal{V},\ q \neq p\}.$$

The elements of $\mathcal{E}$ are called *edges*, and, for $e = \{q,p\} \in \mathcal{E}$, the vertices q,p are called the *endpoints* of e. An *orientation* on $\mathcal{G}$ is defined by two maps $\alpha,\omega : \mathcal{E} \to \mathcal{V}$ such that, for every $e \in \mathcal{E}$, $e = \{\alpha(e),\omega(e)\}$. Given $q,p \in \mathcal{V}$, a *path* from q to p is a n-tuple $(q = q_1,\dots,q_n = p) \in \mathcal{V}^n$ where, for every $j \in [\![1,n-1]\!]$, $\{q_j,q_{j+1}\} \in \mathcal{E}$. The positive integer n is called the *length* of the path. A path of length n in $\mathcal{G}$ is said to be *closed* if $q_1 = q_n$; *simple* if all the edges $\{q_j,q_{j+1}\}$, $j \in [\![1,n-1]\!]$, are different; and *elementary* if the vertices $q_1,\dots,q_n$ are pairwise different, except possibly for the pair (q_1,q_n). An elementary closed path is called a *cycle*. A graph which does not admit cycles is called a *tree*. We say that a graph $\mathcal{G}$ is *connected* if, for every $q,p \in \mathcal{V}$, there exists a path from q to p. We say that $\mathcal{G}$ is *finite* if $\mathcal{V}$ is a finite set. For every $q \in \mathcal{V}$, we denote by $\mathcal{E}_q$ the set of edges for which q is an endpoint, that is,

$$\mathcal{E}_q = \{e \in \mathcal{E} \mid q \in e\}.$$

The cardinality of $\mathcal{E}_q$ is denoted by n_q. We say that $q \in \mathcal{V}$ is *exterior* if $\mathcal{E}_q$ contains at most one element and *interior* otherwise. We denote by $\mathcal{V}_{\mathrm{ext}}$ and $\mathcal{V}_{\mathrm{int}}$ the sets of exterior and interior vertices, respectively. We suppose in the sequel that $\mathcal{V}_{\mathrm{ext}}$ contains at least two elements, and we fix a nonempty subset $\mathcal{V}_{\mathrm{d}}$ of $\mathcal{V}_{\mathrm{ext}}$ such that $\mathcal{V}_{\mathrm{u}} = \mathcal{V}_{\mathrm{ext}} \setminus \mathcal{V}_{\mathrm{d}}$ is nonempty. The vertices of $\mathcal{V}_{\mathrm{d}}$ are said to be *damped*, and the vertices of $\mathcal{V}_{\mathrm{u}}$ are said to be *undamped*. Note that $\mathcal{V}$ is the disjoint union $\mathcal{V} = \mathcal{V}_{\mathrm{int}} \cup \mathcal{V}_{\mathrm{u}} \cup \mathcal{V}_{\mathrm{d}}$.

A *network* is a pair $(\mathcal{G},L)$ where $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is an oriented graph and $L = (L_e)_{e\in\mathcal{E}}$ is a vector of positive real numbers, where each L_e is called the *length* of the edge e. We say that a network is *finite* (respectively, *connected*) if its underlying graph $\mathcal{G}$ is finite (respectively, connected). If $e \in \mathcal{E}$ and $u : [0,L_e] \to \mathbb{C}$ is a function, we write $u(\alpha(e)) = u(0)$ and $u(\omega(e)) = u(L_e)$. For every elementary path $(q_1, \dots, q_n)$, we define its *signature* $s : \mathcal{E} \to \{-1, 0, 1\}$ by

$$s(e) = \begin{cases} 1, & \text{if } e = \{q_i, q_{i+1}\} \text{ for some } i \in [\![1, n-1]\!] \text{ and } \alpha(e) = q_i, \\ -1, & \text{if } e = \{q_i, q_{i+1}\} \text{ for some } i \in [\![1, n-1]\!] \text{ and } \alpha(e) = q_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The *normal derivatives* of u at $\alpha(e)$ and $\omega(e)$ are defined by $\frac{du}{dn_e}(\alpha(e)) = -\frac{du}{dx}(0)$ and $\frac{du}{dn_e}(\omega(e)) = \frac{du}{dx}(L_e)$.

In what follows, we consider only finite connected networks. In order to simplify the notations, we identify $\mathcal{E}$ with the finite set $[\![1, N]\!]$, where $N = \#\mathcal{E}$. We model wave propagation along the edges of a finite connected network $(\mathcal{G}, L)$ by N displacement functions $u_j : [0, +\infty) \times [0, L_j] \to \mathbb{C}$, $j \in [\![1, N]\!]$, satisfying

$$\Sigma_\omega(\mathcal{G}, L, \eta) : \begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(t, x) = \dfrac{\partial^2 u_j}{\partial x^2}(t, x), & j \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_j], \\ u_j(t, q) = u_k(t, q), & q \in \mathcal{V},\ j, k \in \mathcal{E}_q,\ t \in [0, +\infty), \\ \displaystyle\sum_{j \in \mathcal{E}_q} \frac{\partial u_j}{\partial n_j}(t, q) = 0, & q \in \mathcal{V}_{\mathrm{int}},\ t \in [0, +\infty), \\ \dfrac{\partial u_j}{\partial t}(t, q) = -\eta_q(t) \dfrac{\partial u_j}{\partial n_j}(t, q), & q \in \mathcal{V}_{\mathrm{d}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty), \\ u_j(t, q) = 0, & q \in \mathcal{V}_{\mathrm{u}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty). \end{cases} \tag{4.40}$$

Each function η_q is assumed to be nonnegative and determines the damping at the vertex $q \in \mathcal{V}_{\mathrm{d}}$. We denote by η the vector-valued function $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$.

Remark 4.50. Let $(\mathcal{G}, L)$ be a finite connected network with $\mathcal{E} = [\![1, N]\!]$ and (α_1, ω_1), (α_2, ω_2) be two orientations of $\mathcal{G}$. If $(u_j)_{j \in [\![1, N]\!]}$ satisfies (4.40) with orientation (α_1, ω_1) and $(v_j)_{j \in [\![1, N]\!]}$ is given by $v_j = u_j$ if $\alpha_1(j) = \alpha_2(j)$ and $v_j(x) = u_j(L_j - x)$ otherwise, we can easily verify that $(v_j)_{j \in [\![1, N]\!]}$ satisfies (4.40) with orientation (α_2, ω_2). Hence the dynamical properties of (4.40) do not depend on the orientation of $\mathcal{G}$.

For $p \in [1, +\infty]$, consider the Banach spaces $L^p(\mathcal{G}, L) = \prod_{j=1}^N L^p([0, L_j], \mathbb{C})$ and

$$W_0^{1,p}(\mathcal{G}, L) = \left\{ (u_1, \dots, u_N) \in \prod_{j=1}^N W^{1,p}([0, L_j], \mathbb{C}) \,\middle|\, u_j(q) = u_k(q),\ \forall q \in \mathcal{V},\ \forall j, k \in \mathcal{E}_q; u_j(q) = 0,\ \forall q \in \mathcal{V}_{\mathrm{u}},\ \forall j \in \mathcal{E}_q \right\}, \tag{4.41}$$

endowed with the usual norms

$$\|u\|_{L^p(\mathcal{G}, L)} = \begin{cases} \left(\displaystyle\sum_{i=1}^N \|u_i\|_{L^p([0, L_i], \mathbb{C})}^p \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \displaystyle\max_{i \in [\![1, N]\!]} \|u_i\|_{L^\infty([0, L_i], \mathbb{C})}, & \text{if } p = +\infty, \end{cases}$$

$$\|u\|_{W_0^{1,p}(\mathcal{G}, L)} = \begin{cases} \left(\displaystyle\sum_{i=1}^N \left\|u_i'\right\|_{L^p([0, L_i], \mathbb{C})}^p \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \displaystyle\max_{i \in [\![1, N]\!]} \left\|u_i'\right\|_{L^\infty([0, L_i], \mathbb{C})}, & \text{if } p = +\infty. \end{cases}$$

We will omit $(\mathcal{G}, L)$ from the notations when it is clear from the context.

Let $\mathsf{X}_p^\omega = W_0^{1,p} \times L^p$, endowed with the norm $\|\cdot\|_p$ defined by

$$\|(u,v)\|_p = \begin{cases} \left(\|u\|^p_{W_0^{1,p}(\mathcal{G},L)} + \|v\|^p_{L^p(\mathcal{G},L)}\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty), \\ \max\left(\|u\|_{W_0^{1,\infty}(\mathcal{G},L)}, \|v\|_{L^\infty(\mathcal{G},L)}\right), & \text{if } p = +\infty, \end{cases}$$

and, for every $t \in \mathbb{R}$, define the operator $A(t)$ by

$$D(A(t)) = \left\{ (u,v) \in \left(W_0^{1,p} \cap \prod_{j=1}^N W^{2,p}([0,L_j],\mathbb{C})\right) \times W_0^{1,p} \;\middle|\; v_j(q) = -\eta_q(t)\frac{du_j}{dn_j}(q),\ \forall q \in \mathcal{V}_{\mathrm{d}},\ \forall j \in \mathcal{E}_q;\ \sum_{j \in \mathcal{E}_q} \frac{du_j}{dn_j}(q) = 0,\ \forall q \in \mathcal{V}_{\mathrm{int}} \right\},$$

$$A(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}.$$

One can then write (4.40) as an evolution equation in X_p^ω as

$$\dot{U}(t) = A(t)U(t) \tag{4.42}$$

where $U = \left(u, \frac{\partial u}{\partial t}\right)$.

4.4.1 Equivalence with a system of transport equations

In order to make a connection with transport systems, we consider, for $p \in [1,+\infty]$, the Banach space

$$\mathsf{X}_p^\tau = \prod_{j=1}^{2N} L^p([0,L_j^\tau],\mathbb{C}),$$

where $L_{2j-1}^\tau = L_{2j}^\tau = L_j$ for $j \in [\![1,N]\!]$.

Definition 4.51 (D'Alembert decomposition operator)**.** Let $T : \mathsf{X}_p^\omega \to \mathsf{X}_p^\tau$ be the operator given by $T(u,v) = f$, where, for $j \in [\![1,N]\!]$, $x \in [0,L_j]$,

$$f_{2j-1}(x) = u_j'(L_j - x) + v_j(L_j - x), \qquad f_{2j}(x) = u_j'(x) - v_j(x). \tag{4.43}$$

In order to describe the range of T, we introduce the following notations. Let $r \in \mathbb{N}$ be the number of elementary paths $(q_1,\dots,q_n)$ in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_u$. The set of such paths will be indexed by $[\![1,r]\!]$. We denote by s_i the signature of the path corresponding to the index $i \in [\![1,r]\!]$. We define $R = (\rho_{ij})_{i,j} \in \mathcal{M}_{r,2N}(\mathbb{C})$ by setting

$$\rho_{i,2j-1} = \rho_{i,2j} = s_i(j) \text{ for } i \in [\![1,r]\!],\ j \in [\![1,N]\!].$$

We then have the following proposition.

Proposition 4.52. *The operator T is a bijection from X_p^ω to $\mathsf{Y}_p(R)$. Moreover, T and T^{-1} are continuous.*

Proof. Let $(u,v) \in \mathsf{X}_p^\omega$ and let $f = T(u,v) \in \mathsf{X}_p^\tau$. Let $(q_1, \dots, q_n)$ be an elementary path in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_{\mathrm{u}}$ and let s be its signature. For $i \in [\![1, n-1]\!]$, let j_i be the index corresponding to the edge $\{q_i, q_{i+1}\}$. We have

$$\begin{aligned}\sum_{j=1}^N s(j) \int_0^{L_j} \left(f_{2j-1}(x) + f_{2j}(x)\right) dx &= 2 \sum_{j=1}^N s(j) \int_0^{L_j} u_j'(x) dx = 2 \sum_{i=1}^N s(j) \left(u_j(L_j) - u_j(0)\right) \\ &= 2 \sum_{i=1}^{n-1} \left(u_{j_i}(q_{i+1}) - u_{j_i}(q_i)\right) = 2\left(u_{j_{n-1}}(q_n) - u_{j_1}(q_1)\right) = 0,\end{aligned}$$

and thus $f \in \mathsf{Y}_p(R)$.

Conversely, take $f \in \mathsf{Y}_p(R)$. For $j \in [\![1, N]\!]$, define $v_j : [0, L_j] \to \mathbb{C}$ by

$$v_j(x) = \frac{f_{2j-1}(L_j - x) - f_{2j}(x)}{2}. \tag{4.44}$$

One clearly has $v_j \in L^p([0, L_j], \mathbb{C})$. We define u_j as follows: let $e \in \mathcal{E}$ be the edge corresponding to the index j. Let $(q_1, \dots, q_n)$ be any elementary path with $q_1 \in \mathcal{V}_{\mathrm{u}}$ and $q_n = \alpha(e)$. Let $s : \mathcal{E} \to \{-1, 0, 1\}$ be the signature of that path and, for $i \in [\![1, n-1]\!]$, let j_i be the index associated with the edge $\{q_i, q_{i+1}\}$. For $x \in [0, L_j]$, set

$$u_j(x) = \sum_{i=1}^{n-1} s(j_i) \int_0^{L_{j_i}} \frac{f_{2j_i - 1}(\xi) + f_{2j_i}(\xi)}{2} d\xi + \int_0^x \frac{f_{2j-1}(L_j - \xi) + f_{2j}(\xi)}{2} d\xi. \tag{4.45}$$

This definition does not depend on the choice of the path $(q_1, \dots, q_n)$ with $q_1 \in \mathcal{V}_{\mathrm{u}}$ and $q_n = \alpha(e)$ thanks to the definition of the matrix R. It is an immediate consequence of (4.45) that $(u,v) \in \mathsf{X}_p^\omega$. The map $f \mapsto (u,v)$ defines an operator $S : \mathsf{Y}_p(R) \to \mathsf{X}_p^\omega$. We clearly have $T \circ S = \mathrm{Id}_{\mathsf{Y}_p(R)}$ and $S \circ T = \mathrm{Id}_{\mathsf{X}_p^\omega}$, and thus T is bijective. The continuity of T and S follows immediately from (4.43), (4.44), and (4.45). ∎

Remark 4.53. When $p = 2$, one easily checks that $\frac{1}{\sqrt{2}} T : \mathsf{X}_2^\omega \to \mathsf{Y}_2(R)$ is unitary.

Remark 4.54. The operator T corresponds to the d'Alembert decomposition of the solutions of the one-dimensional wave equation into a pair of traveling waves moving in opposite directions. For every $j \in [\![1, N]\!]$, f_{2j-1} and f_{2j} correspond to the waves moving from $\omega(j)$ to $\alpha(j)$ and from $\alpha(j)$ to $\omega(j)$, respectively (see Figure 4.1).

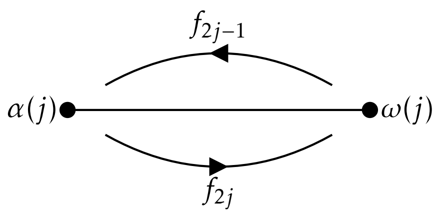


Figure 4.1: D'Alembert decomposition of the wave equation on the edge $j \in [\![1, N]\!]$.

Let us consider the operator $B(t)$ in $\mathsf{Y}_p(R)$ defined by conjugation as

$$D(B(t)) = \{f \in \mathsf{Y}_p(R) \mid T^{-1} f \in D(A(t))\}, \quad B(t) f = T A(t) T^{-1} f.$$

In order to give a more explicit formula for $B(t)$, we introduce the following notations.

Definition 4.55 (Inward and outward decompositions)**.** The *inward and outward decompositions* of $\mathbb{C}^{2N}$ are defined respectively as the direct sums

$$\mathbb{C}^{2N} = \bigoplus_{q \in \mathcal{V}} W_{\mathrm{in}}^{q}, \qquad \mathbb{C}^{2N} = \bigoplus_{q \in \mathcal{V}} W_{\mathrm{out}}^{q},$$

where, for every $q \in \mathcal{V}$, we set

$$\begin{aligned} W_{\mathrm{in}}^{q} &= \operatorname{Span}\Big(\{e_{2j} \mid \omega(j) = q\} \cup \{e_{2j-1} \mid \alpha(j) = q\}\Big), \\ W_{\mathrm{out}}^{q} &= \operatorname{Span}\Big(\{e_{2j} \mid \alpha(j) = q\} \cup \{e_{2j-1} \mid \omega(j) = q\}\Big). \end{aligned}$$

For every $q \in \mathcal{V}$, we denote by Π_{in}^{q} and Π_{out}^{q} the canonical projections of $\mathbb{C}^{2N}$ onto W_{in}^{q} and W_{out}^{q}, respectively, which we identify with matrices in $\mathcal{M}_{n_q,2N}(\mathbb{C})$.

For $n \in \mathbb{N}$, let J_n denote the $n \times n$ matrix with all elements equal to 1. Define $D = \operatorname{diag}((-1)^j)_{j \in [\![1,2N]\!]}$. For $q \in \mathcal{V}$ and $t \in \mathbb{R}$, we set

$$M^q(t) = \begin{cases} \left(\Pi_{\mathrm{out}}^{q}\right)^{\mathrm{T}} \left(\mathrm{Id}_{n_q} - \dfrac{2}{n_q} J_{n_q}\right) \Pi_{\mathrm{in}}^{q}, & \text{if } q \in \mathcal{V}_{\mathrm{int}}, \\ \left(\Pi_{\mathrm{out}}^{q}\right)^{\mathrm{T}} \Pi_{\mathrm{in}}^{q}, & \text{if } q \in \mathcal{V}_{\mathrm{u}}, \\ \dfrac{1 - \eta_q(t)}{1 + \eta_q(t)} \left(\Pi_{\mathrm{out}}^{q}\right)^{\mathrm{T}} \Pi_{\mathrm{in}}^{q}, & \text{if } q \in \mathcal{V}_{\mathrm{d}}. \end{cases}$$

We define the time-dependent matrix $M = (m_{ij})_{i,j \in [\![1,2N]\!]}$ by

$$M = -D \left(\sum_{q \in \mathcal{V}} M^q\right) D. \tag{4.46}$$

Remark 4.56. If the components of η are nonnegative measurable functions, then M is measurable and its components take values in $[-1, 1]$.

Remark 4.57. Notice that $W_{\mathrm{in}}^{q_1}$ and $W_{\mathrm{in}}^{q_2}$ are orthogonal whenever $q_1 \neq q_2$, and similarly for the outward decomposition. Moreover, for each $q \in \mathcal{V}$, the spaces W_{in}^{q} and W_{out}^{q} are invariant under D. We finally notice that the image of $M^q(t)$ is contained in W_{out}^{q}. From these observations, we deduce that, for every $q \in \mathcal{V}$ and $t \in \mathbb{R}$,

$$\Pi_{\mathrm{out}}^{q} D M(t) = -\Pi_{\mathrm{out}}^{q} M^q(t) D.$$

We finally obtain the following expression for $B(t)$.

Proposition 4.58. *For $t \in \mathbb{R}$ and $p \in [1, +\infty]$, the operator $B(t)$ is given by*

$$D(B(t)) = \left\{ f \in \mathsf{Y}_p(R) \cap \prod_{i=1}^{2N} W^{1,p}([0, L_i^{\tau}], \mathbb{C}) \;\middle|\; f_i(0) = \sum_{j=1}^{2N} m_{ij}(t) f_j(L_j^{\tau}),\ \forall i \in [\![1, 2N]\!] \right\}, \tag{4.47}$$

$$B(t) f = -f'. \tag{4.48}$$

Proof. Let $f \in \mathsf{Y}_p(R)$ and $(u, v) = T^{-1} f \in \mathsf{X}_p^{\omega}$ and notice that

$$u_j'(x) = \frac{f_{2j-1}(L_j - x) + f_{2j}(x)}{2}, \qquad v_j(x) = \frac{f_{2j-1}(L_j - x) - f_{2j}(x)}{2}. \tag{4.49}$$

It follows from (4.43) and (4.49) that $f_i \in W^{1,p}([0, L_i^\tau], \mathbb{C})$ for every $i \in [\![1, 2N]\!]$ if and only if $u_i \in W^{2,p}([0, L_i], \mathbb{C})$ and $v_i \in W^{1,p}([0, L_i], \mathbb{C})$ for every $i \in [\![1, N]\!]$.

We suppose from now on that $f_i \in W^{1,p}([0, L_i^\tau], \mathbb{C})$ for every $i \in [\![1, 2N]\!]$. Take $F_0 = (f_i(0))_{i \in [\![1,2N]\!]}$ and $F_L = (f_i(L_i^\tau))_{i \in [\![1,2N]\!]}$. The condition

$$f_i(0) = \sum_{j=1}^{2N} m_{ij}(t) f_j(L_j^\tau), \qquad \forall i \in [\![1, 2N]\!] \tag{4.50}$$

can be written as $F_0 = M(t) F_L$. Thanks to the outward decomposition of $\mathbb{C}^{2N}$, this is equivalent to $\Pi_{\mathrm{out}}^q D F_0 = \Pi_{\mathrm{out}}^q D M(t) F_L$ for every $q \in \mathcal{V}$. By Remark 4.57, we have $\Pi_{\mathrm{out}}^q D M(t) = -\Pi_{\mathrm{out}}^q M^q(t) D$, and thus (4.50) is equivalent to

$$\Pi_{\mathrm{out}}^q D F_0 + \Pi_{\mathrm{out}}^q M^q(t) D F_L = 0, \qquad \forall q \in \mathcal{V}. \tag{4.51}$$

If $q \in \mathcal{V}_{\mathrm{d}}$, let j be the index corresponding to the unique edge in $\mathcal{E}_q$. To simplify the notations, we consider here the case $\alpha(j) = q$, the other case being analogous. Then

$$\begin{aligned}
\Pi_{\mathrm{out}}^q D F_0 + \Pi_{\mathrm{out}}^q M^q(t) D F_L &= \Pi_{\mathrm{out}}^q D F_0 + \frac{1-\eta_q(t)}{1+\eta_q(t)} \Pi_{\mathrm{in}}^q D F_L \\
= & f_{2j}(0) - \frac{1-\eta_q(t)}{1+\eta_q(t)} f_{2j-1}(L_j) = u_j'(0) - v_j(0) - \frac{1-\eta_q(t)}{1+\eta_q(t)} \left(u_j'(0) + v_j(0)\right) \\
= & \frac{2}{1+\eta_q(t)} \left(\eta_q(t) u_j'(0) - v_j(0)\right),
\end{aligned}$$

which shows that the left-hand side is equal to zero if and only if one has $v_j(q) = -\eta_q(t) \frac{du_j}{dn_j}(q)$. If $q \in \mathcal{V}_{\mathrm{u}}$, the same argument shows that the left-hand side is equal to zero if and only if $v_j(q) = 0$.

Finally, if $q \in \mathcal{V}_{\mathrm{int}}$, one easily obtains that

$$\Pi_{\mathrm{in}}^q D F_L = \left(\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q}, \qquad \Pi_{\mathrm{out}}^q D F_0 = \left(-\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q}.$$

Since $\Pi_{\mathrm{out}}^q \left(\Pi_{\mathrm{out}}^q\right)^{\mathrm{T}} = \mathrm{Id}_{W_{\mathrm{out}}^q}$, one has

$$\begin{aligned}
\Pi_{\mathrm{out}}^q D F_0 + \Pi_{\mathrm{out}}^q M^q(t) D F_L &= \left(-\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q} + \left(\mathrm{Id}_{n_q} - \frac{2}{n_q} J_{n_q}\right) \left(\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q} \\
&= \left(-2 v_j(q) - \tfrac{2}{n_q} \textstyle\sum_{k \in \mathcal{E}_q} \left(\tfrac{du_k}{dn_k}(q) - v_k(q)\right)\right)_{j \in \mathcal{E}_q}.
\end{aligned}$$

The right-hand side is equal to zero if and only if $v_j(q) = v_k(q)$ for every $j, k \in \mathcal{E}_q$ and $\sum_{k \in \mathcal{E}_q} \frac{du_k}{dn_k}(q) = 0$.

Collecting all the equivalences corresponding to the identities in (4.51), we conclude that (4.47) holds.

Let now $f \in D(B(t))$ and denote $(u, v) = T^{-1} f \in D(A(t))$, $g = B(t) f$. Then

$$g = T A(t) T^{-1} f = T A(t)(u, v) = T(v, u''),$$

and so, by (4.43), for every $j \in [\![1, 2N]\!]$,

$$\begin{aligned}
g_{2j-1}(x) &= v_j'(L_j - x) + u_j''(L_j - x) = -\frac{d}{dx}\left(v_j(L_j - x) + u_j'(L_j - x)\right) = -f_{2j-1}'(x), \\
g_{2j}(x) &= v_j'(x) - u_j''(x) = \frac{d}{dx}\left(v_j(x) - u_j'(x)\right) = -f_{2j}'(x),
\end{aligned}$$

which shows that (4.48) holds. ■

The operator $T : \mathsf{X}_p^\omega \to \mathsf{Y}_p(R)$ transforms (4.42) into

$$\dot{F}(t) = B(t) F(t).$$

This evolution equation corresponds to the system of transport equations

$$\begin{cases} \dfrac{\partial f_i}{\partial t}(t,x) + \dfrac{\partial f_i}{\partial x}(t,x) = 0, & i \in [\![1, 2N]\!],\ t \in [0, +\infty),\ x \in [0, L_i^\tau], \\ f_i(t,0) = \displaystyle\sum_{j=1}^{2N} m_{ij}(t) f_j(t, L_j^\tau), & i \in [\![1, 2N]\!],\ t \in [0, +\infty), \end{cases} \tag{4.52}$$

where $F(t) = (f_i(t))_{i \in [\![1,2N]\!]}$. The following property of the matrix $M(t)$ will be useful in the sequel.

Lemma 4.59. *For every $t \in \mathbb{R}$,*

$$M(t)^{\mathrm{T}} M(t) = \mathrm{Id}_{2N} - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \frac{4\eta_q(t)}{(1+\eta_q(t))^2} (\Pi_{\mathrm{in}}^q)^{\mathrm{T}} \Pi_{\mathrm{in}}^q.$$

Proof. Notice that, for every $q \in \mathcal{V}$, $M^q(t)$ can be written as

$$M^q(t) = (\Pi_{\mathrm{out}}^q)^{\mathrm{T}} \left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right) \Pi_{\mathrm{in}}^q,$$

where $\lambda_q(t) = \frac{1-\eta_q(t)}{1+\eta_q(t)}$ if $q \in \mathcal{V}_{\mathrm{d}}$ and $\lambda_q(t) = 1$ otherwise, while $\delta_q = 1$ if $q \in \mathcal{V}_{\mathrm{int}}$ and $\delta_q = 0$ otherwise. By a straightforward computation, one verifies that, for every $q \in \mathcal{V}$,

$$\left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right)^{\mathrm{T}} \left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right) = \lambda_q(t)^2 \mathrm{Id}_{n_q}.$$

Noticing furthermore that, for every $q_1, q_2 \in \mathcal{V}$, $\Pi_{\mathrm{out}}^{q_1} (\Pi_{\mathrm{out}}^{q_2})^{\mathrm{T}} = \delta_{q_1 q_2} \mathrm{Id}_{W_{\mathrm{out}}^{q_1}}$, one deduces that

$$M(t)^{\mathrm{T}} M(t) = D \left[\sum_{q \in \mathcal{V}} \lambda_q(t)^2 (\Pi_{\mathrm{in}}^q)^{\mathrm{T}} \Pi_{\mathrm{in}}^q\right] D.$$

Since the term between brackets in the above equation is diagonal and $\lambda_q(t)^2 = 1 - \frac{4\eta_q(t)}{(1+\eta_q(t))^2}$ for $q \in \mathcal{V}_{\mathrm{d}}$, the conclusion follows. ■

4.4.2 Existence of solutions

Thanks to the operator $T : \mathsf{X}_p^\omega \to \mathsf{Y}_p(R)$, one can give the following definition for solutions of (4.40).

Definition 4.60. Let $U_0 \in \mathsf{X}_p^\omega$ and $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$ be a measurable function with nonnegative components. We say that $U : \mathbb{R}_+ \to \mathsf{X}_p^\omega$ is a *solution* of $\Sigma_\omega(\mathcal{G}, L, \eta)$ with initial condition U_0 if $T^{-1} U : \mathbb{R}_+ \to \mathsf{Y}_p(R)$ is a solution of (4.52) with initial condition $T^{-1} U_0 \in \mathsf{Y}_p(R)$.

For every $F_0 \in \mathsf{Y}_p(R)$, it follows from Proposition 4.40 that (4.52) admits a unique solution $F : \mathbb{R}_+ \to \mathsf{X}_p^\tau$. In order to show that this solution remains in $\mathsf{Y}_p(R)$ for every $t \geq 0$, one needs to show that $\mathsf{Y}_p(R)$ is invariant under the flow of (4.52).

Proposition 4.61. *For every $t \in \mathbb{R}$, $RM(t) = R$.*

Proof. Thanks to the inward decomposition of $\mathbb{C}^{2N}$, we prove the proposition by showing that for every $q \in \mathcal{V}$ and $t \in \mathbb{R}$,

$$-RD(\Pi_{\mathrm{out}}^{q})^{\mathrm{T}}\left[\lambda_q(t)\mathrm{Id}_{n_q} - \frac{2}{n_q}\delta_q J_{n_q}\right] = RD(\Pi_{\mathrm{in}}^{q})^{\mathrm{T}}, \tag{4.53}$$

where $\lambda_q(t)$ and δ_q are defined as in the proof of Lemma 4.59. Without loss of generality, it is enough to consider the case where R is a line matrix, i.e., we consider a single elementary path $(q_1,\dots,q_n)$ in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_{\mathrm{u}}$, with signature s. Then $R = (\rho_j)_{j\in[\![1,2N]\!]}$ is given by $\rho_{2j-1} = \rho_{2j} = s(j)$ for $j \in [\![1,N]\!]$. For $i \in [\![1,n-1]\!]$, denote by j_i the edge corresponding to $\{q_i, q_{i+1}\}$. Let us write $R = \sum_{i=1}^{n-1} s(j_i)(e_{2j_i-1} + e_{2j_i})^{\mathrm{T}}$ and notice that

$$RD = \sum_{i=1}^{n-1} s(j_i)(-e_{2j_i-1} + e_{2j_i})^{\mathrm{T}}.$$

By definition of the signature s, one has, for $i \in [\![1,n-1]\!]$,

$$\begin{aligned} -s(j_i)e_{2j_i-1}^{\mathrm{T}} &= e_{2j_i-1}^{\mathrm{T}}\left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_i}\right], \\ s(j_i)e_{2j_i}^{\mathrm{T}} &= e_{2j_i}^{\mathrm{T}}\left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_i}\right], \end{aligned}$$

and

$$\begin{aligned} -s(j_i)e_{2j_i-1}^{\mathrm{T}} &= e_{2j_i-1}^{\mathrm{T}}\left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_{i+1}}\right], \\ s(j_i)e_{2j_i}^{\mathrm{T}} &= e_{2j_i}^{\mathrm{T}}\left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_{i+1}}\right]. \end{aligned}$$

One deduces that

$$\begin{aligned} RD &= \sum_{i=1}^{n-1}(e_{2j_i-1} + e_{2j_i})^{\mathrm{T}}\left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{in}}^{q_i}\right] \\ &= \sum_{i=1}^{n-1}(e_{2j_i-1} + e_{2j_i})^{\mathrm{T}}\left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}}\Pi_{\mathrm{out}}^{q_{i+1}}\right]. \end{aligned}$$

By using the above relations, Equation (4.53) can be rewritten as

$$\begin{aligned} &\left[\lambda_q(t)\mathrm{Id}_{n_q} - \frac{2}{n_q}\delta_q J_{n_q}\right]\Pi_{\mathrm{out}}^{q}\sum_{i=1}^{n-1}\left(\delta_{qq_{i+1}} - \delta_{qq_i}\right)(e_{2j_i-1} + e_{2j_i}) \\ &\qquad\qquad = \Pi_{\mathrm{in}}^{q}\sum_{i=1}^{n-1}\left(\delta_{qq_{i+1}} - \delta_{qq_i}\right)(e_{2j_i-1} + e_{2j_i}). \end{aligned} \tag{4.54}$$

Such an identity is trivially satisfied if $q \notin \{q_1,\dots,q_n\}$. Assume now that either $q = q_i$ for some $i \in [\![2,n-1]\!]$ or $q = q_1 = q_n$ (and in the latter case set $i = n$ and define $j_{n+1} = j_1$). In particular, $q \in \mathcal{V}_{\mathrm{int}}$ and $\lambda_q(t) = \delta_q = 1$. We therefore must prove that

$$\left[\mathrm{Id}_{n_{q_i}} - \frac{2}{n_{q_i}}J_{n_{q_i}}\right]\Pi_{\mathrm{out}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}) = \Pi_{\mathrm{in}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}). \tag{4.55}$$

By definition of $\Pi_{\mathrm{in}}^{q_i}$ and $\Pi_{\mathrm{out}}^{q_i}$, one has that

$$\Pi_{\mathrm{out}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}) = \Pi_{\mathrm{in}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}) = w,$$

where $w \in \mathbb{C}^{n_{q_i}}$ has all its coordinates equal to zero, except two of them, one equal to 1 and the other one equal to -1. Hence $J_{n_{q_i}} w = 0$ and (4.55) holds true.

It remains to treat the case $q \in \{q_1, q_n\} \subset \mathcal{V}_{\mathrm{u}}$. In this case, $\lambda_q(t) = 1$ and $\delta_q = 0$, and we furthermore assume, with no loss of generality, that $q = q_1$. We can rewrite (4.54) as

$$\Pi_{\mathrm{out}}^{q_1}(e_{2j_1-1} + e_{2j_1}) = \Pi_{\mathrm{in}}^{q_1}(e_{2j_1-1} + e_{2j_1}),$$

which holds true by definition of $\Pi_{\mathrm{in}}^{q_1}$ and $\Pi_{\mathrm{out}}^{q_1}$. ■

The main result of the section, given next, follows immediately from Propositions 4.42 and 4.61.

Proposition 4.62. *Let $(\mathcal{G}, L)$ be a network, $p \in [1, +\infty]$, and $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$ be a measurable function with nonnegative components. Then, for every $U_0 \in \mathsf{X}_p^{\omega}$, the system $\Sigma_{\omega}(\mathcal{G}, L, \eta)$ defined in (4.40) admits a unique solution $U : \mathbb{R}_+ \to \mathsf{X}_p^{\omega}$.*

4.4.3 Stability of solutions

We next provide an appropriate definition of exponential type for (4.40).

Definition 4.63. Let $(\mathcal{G}, L)$ be a network, $p \in [1, +\infty]$, and $\mathcal{D}$ be a subset of the space of measurable functions $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$ with nonnegative components. Denote by $\Sigma_{\omega}(\mathcal{G}, L, \mathcal{D})$ the family of systems $\Sigma_{\omega}(\mathcal{G}, L, \eta)$ for $\eta \in \mathcal{D}$. We say that $\Sigma_{\omega}(\mathcal{G}, L, \mathcal{D})$ is of *exponential type* γ in X_p^{ω} if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $\eta \in \mathcal{D}$ and $u_0 \in \mathsf{X}_p^{\omega}$, the corresponding solution u of $\Sigma_{\omega}(\mathcal{G}, L, \eta)$ satisfies, for every $t \geq 0$,

$$\|u(t)\|_p \leq K e^{(\gamma+\varepsilon)t} \|u_0\|_p.$$

We say that $\Sigma_{\omega}(\mathcal{G}, L, \mathcal{D})$ is *exponentially stable* in X_p^{ω} if it is of negative exponential type.

Given $\mathcal{D}$ as in the above definition, we define

$$\mathcal{M} = \{M : \mathbb{R} \to \mathcal{M}_{2N}(\mathbb{R}) \mid M \text{ is given by (4.46) for some } \eta \in \mathcal{D}\}.$$

Thanks to the continuity of T and T^{-1} established in Proposition 4.52, we remark that $\Sigma_{\omega}(\mathcal{G}, L, \mathcal{D})$ is of exponential type γ in X_p^{ω} if and only if $\Sigma_{\tau}(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$. As a consequence of Corollary 4.48, we have the following result in the case of arbitrarily switching dampings η_q, $q \in \mathcal{V}_{\mathrm{d}}$.

Corollary 4.64. *Let $(\mathcal{G}, \Lambda)$ be a network, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D}$ be a subset of $(\mathbb{R}_+)^d$, and $\mathcal{D} = L^{\infty}(\mathbb{R}, \mathfrak{D})$. The following statements are equivalent.*

(a) *$\Sigma_{\omega}(\mathcal{G}, \Lambda, \mathcal{D})$ is exponentially stable in X_p^{ω} for some $p \in [1, +\infty]$.*

(b) *$\Sigma_{\omega}(\mathcal{G}, L, \mathcal{D})$ is exponentially stable in X_p^{ω} for every $L \in V_+(\Lambda)$ and $p \in [1, +\infty]$.*

We can now provide a necessary and sufficient condition on $\mathcal{G}$ and $\mathfrak{D}$ for the exponential stability of $\Sigma_{\omega}(\mathcal{G}, \Lambda, L^{\infty}(\mathbb{R}, \mathfrak{D}))$.

Theorem 4.65. *Let $(\mathcal{G}, \Lambda)$ be a network, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D}$ be a bounded subset of $(\mathbb{R}_+)^d$, and $\mathcal{D} = L^{\infty}(\mathbb{R}, \mathfrak{D})$. Then $\Sigma_{\omega}(\mathcal{G}, \Lambda, \mathcal{D})$ is exponentially stable in X_p^{ω} for some $p \in [1, +\infty]$ if and only if $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one vertex, and $\overline{\mathfrak{D}} \subset (\mathbb{R}_+^*)^d$.*

Corollary 4.64 allows one to easily prove the "only if" part of Theorem 4.65. The "if" part follows from a standard energy estimate and an observability inequality, which can be obtained as in [63, Chapter 4, Section 4.1] (see also [155]). For the sake of completeness, we provide here a complete proof of Theorem 4.65, the "if" part being proved following the arguments from [63, 155]. We start with a few preliminary results needed for the energy estimates in the "if" part.

For $U = (u, v)$ a solution of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ in X_2^ω, $t \in \mathbb{R}_+$, and $j \in [\![1, N]\!]$, we define the energy

$$E_j(t) = \int_0^{\Lambda_j} \left(\left| \frac{\partial u_j}{\partial x}(t, x) \right|^2 + \left| v_j(t, x) \right|^2 \right) dx. \tag{4.56}$$

In particular, $\|U(t)\|_2^2 = \sum_{j=1}^N E_j(t)$. Notice that, by setting $f = \frac{1}{\sqrt{2}} T U$, one has $\|f(t)\|_{\mathsf{Y}_2(R)} = \|U(t)\|_2$ thanks to Remark 4.53 and

$$E_j(t) = \int_0^{\Lambda_j} \left(\left| f_{2j-1}(t, x) \right|^2 + \left| f_{2j}(t, x) \right|^2 \right) dx.$$

Lemma 4.66. *Let $U = (u, v)$ be a solution of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ in X_2^ω and $f = \frac{1}{\sqrt{2}} T U$. Then, for every $j \in [\![1, N]\!]$ and $t \geq \Lambda_j$, the energy (4.56) satisfies*

$$E_j(t) \leq \int_{t-\Lambda_j}^{t+\Lambda_j} \left(\left| f_{2j-1}(\tau, 0) \right|^2 + \left| f_{2j}(\tau, \Lambda_j) \right|^2 \right) d\tau. \tag{4.57}$$

Proof. We have

$$\begin{aligned} E_j(t) &= \int_0^{\Lambda_j} \left(\left| f_{2j-1}(t, x) \right|^2 + \left| f_{2j}(t, \Lambda_j - x) \right|^2 \right) dx = \int_0^{\Lambda_j} \left(\left| f_{2j-1}(t - x, 0) \right|^2 + \left| f_{2j}(t + x, \Lambda_j) \right|^2 \right) dx \\ &= \int_{t-\Lambda_j}^{t} \left| f_{2j-1}(\tau, 0) \right|^2 d\tau + \int_t^{t+\Lambda_j} \left| f_{2j}(\tau, \Lambda_j) \right|^2 d\tau \\ &\leq \int_{t-\Lambda_j}^{t+\Lambda_j} \left(\left| f_{2j-1}(\tau, 0) \right|^2 + \left| f_{2j}(\tau, \Lambda_j) \right|^2 \right) d\tau. \end{aligned}$$

■

Lemma 4.67. *Let $U = (u, v)$ be a solution of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ in X_2^ω and $f = \frac{1}{\sqrt{2}} T U$. Let $q \in \mathcal{V}_{\mathrm{int}}$, $j \in \mathcal{E}_q$, and suppose that $\omega(j) = q$ and $\alpha(i) = q$ for every $i \in \mathcal{E}_q \setminus \{j\}$. For every $a > 0$ and $t \geq a + \max_{i \in \mathcal{E}_q \setminus \{j\}} \Lambda_i$, we have*

$$\int_{t-a}^{t+a} \left(\left| f_{2j-1}(\tau, 0) \right|^2 + \left| f_{2j}(\tau, \Lambda_j) \right|^2 \right) d\tau \leq 4(n_q - 1) \sum_{i \in \mathcal{E}_q \setminus \{j\}} \int_{t-a-\Lambda_i}^{t+a+\Lambda_i} \left(|f_{2i-1}(\tau, 0)|^2 + |f_{2i}(\tau, \Lambda_i)|^2 \right) d\tau.$$

Proof. Thanks to (4.46), we have

$$\begin{aligned} f_{2j-1}(\tau, 0) &= \frac{n_q - 2}{n_q} f_{2j}(\tau, \Lambda_j) + \frac{2}{n_q} \sum_{i \in \mathcal{E}_q \setminus \{j\}} f_{2i-1}(\tau, \Lambda_i), \\ f_{2j}(\tau, \Lambda_j) &= \frac{n_q - 2}{n_q} f_{2j-1}(\tau, 0) + \frac{2}{n_q} \sum_{i \in \mathcal{E}_q \setminus \{j\}} f_{2i}(\tau, 0). \end{aligned}$$

Hence

$$\begin{aligned} f_{2j-1}(\tau, 0) &= \frac{n_q}{2(n_q - 1)} \sum_{i \in \mathcal{E}_q \setminus \{j\}} \left[f_{2i-1}(\tau, \Lambda_i) + \frac{n_q - 2}{n_q} f_{2i}(\tau, 0) \right], \\ f_{2j}(\tau, \Lambda_j) &= \frac{n_q}{2(n_q - 1)} \sum_{i \in \mathcal{E}_q \setminus \{j\}} \left[f_{2i}(\tau, 0) + \frac{n_q - 2}{n_q} f_{2i-1}(\tau, \Lambda_i) \right]. \end{aligned}$$

We finally obtain

$$\begin{aligned}
&\int_{t-a}^{t+a}\left(\left|f_{2j-1}(\tau,0)\right|^2+\left|f_{2j}(\tau,\Lambda_j)\right|^2\right)d\tau\\
&\quad=\frac{n_q^2}{4(n_q-1)^2}\int_{t-a}^{t+a}\left(\left|\sum_{i\in\mathcal{E}_q\setminus\{j\}}\left[f_{2i-1}(\tau,\Lambda_i)+\frac{n_q-2}{n_q}f_{2i}(\tau,0)\right]\right|^2\right.\\
&\qquad\left.+\left|\sum_{i\in\mathcal{E}_q\setminus\{j\}}\left[f_{2i}(\tau,0)+\frac{n_q-2}{n_q}f_{2i-1}(\tau,\Lambda_i)\right]\right|^2\right)d\tau\\
&\quad\leq 4(n_q-1)\sum_{i\in\mathcal{E}_q\setminus\{j\}}\int_{t-a}^{t+a}\left(|f_{2i-1}(\tau,\Lambda_i)|^2+|f_{2i}(\tau,0)|^2\right)d\tau\\
&\quad=4(n_q-1)\sum_{i\in\mathcal{E}_q\setminus\{j\}}\left(\int_{t-a-\Lambda_i}^{t+a-\Lambda_i}|f_{2i-1}(\tau,0)|^2\,d\tau+\int_{t-a+\Lambda_i}^{t+a+\Lambda_i}|f_{2i}(\tau,\Lambda_i)|^2\,d\tau\right)\\
&\qquad\qquad\leq 4(n_q-1)\sum_{i\in\mathcal{E}_q\setminus\{j\}}\int_{t-a-\Lambda_i}^{t+a+\Lambda_i}\left(|f_{2i-1}(\tau,0)|^2+|f_{2i}(\tau,\Lambda_i)|^2\right)d\tau. \qquad\blacksquare
\end{aligned}$$

Lemma 4.68. *Let $U=(u,v)$ be a solution of $\Sigma_\omega(\mathcal{G},\Lambda,\mathcal{D})$ in X_2^ω and $f=\frac{1}{\sqrt{2}}TU$. Let $q\in\mathcal{V}_{\mathrm{d}}$, $j\in\mathcal{E}_q$, and suppose that $\omega(j)=q$. For every $a>0$ and $t\geq a$, we have*

$$\int_{t-a}^{t+a}\left(\left|f_{2j-1}(\tau,0)\right|^2+\left|f_{2j}(\tau,\Lambda_j)\right|^2\right)d\tau\leq 2\int_{t-a}^{t+a}\left|f_{2j}(\tau,\Lambda_j)\right|^2d\tau.$$

Proof. Since $f_{2j-1}(\tau,0)=\frac{1-\eta_q(\tau)}{1+\eta_q(\tau)}f_{2j}(\tau,\Lambda_j)$, we get

$$\begin{aligned}
\int_{t-a}^{t+a}\left(\left|f_{2j-1}(\tau,0)\right|^2+\left|f_{2j}(\tau,\Lambda_j)\right|^2\right)d\tau&=\int_{t-a}^{t+a}\left(\left|\frac{1-\eta_q(\tau)}{1+\eta_q(\tau)}\right|^2+1\right)\left|f_{2j}(\tau,\Lambda_j)\right|^2d\tau\\
&\leq 2\int_{t-a}^{t+a}\left|f_{2j}(\tau,\Lambda_j)\right|^2d\tau. \qquad\blacksquare
\end{aligned}$$

We can now turn to the proof of Theorem 4.65.

Proof of Theorem 4.65. Similarly to Remark 4.49, the exponential stability of $\Sigma_\omega(\mathcal{G},\Lambda,\mathcal{D})$ is equivalent to that of $\Sigma_\omega(\mathcal{G},\Lambda,L^\infty(\mathbb{R},\overline{\mathcal{D}}))$. We therefore assume with no loss of generality that $\mathcal{D}$ is compact.

Suppose that either $\mathcal{G}$ is not a tree, $\mathcal{V}_{\mathrm{u}}$ contains more than one vertex, or $\mathcal{D}$ contains a point $\overline{\eta}$ with $\overline{\eta}_{\overline{q}}=0$ for some $\overline{q}\in\mathcal{V}_{\mathrm{d}}$. Let $(q_1,\dots,q_n)$ be an elementary path in $\mathcal{G}$ with $q_1=q_n$, $q_1,q_n\in\mathcal{V}_{\mathrm{u}}$, or $q_1\in\mathcal{V}_{\mathrm{u}}$ and $q_n=\overline{q}$. Let s be its signature and, for $i\in[\![1,n-1]\!]$, let j_i be the index corresponding to the edge $\{q_i,q_{i+1}\}$. Take $L\in V_+(\Lambda)\cap\mathbb{N}^N$, which is possible thanks to Proposition 4.9. For $j\in[\![1,N]\!]$, we define

$$u_j(t,x)=\begin{cases}s(j_i)\sin(2\pi t)\sin(2\pi x), & \text{if } j=j_i \text{ for a certain } i\in[\![1,n-1]\!],\\ 0, & \text{otherwise.}\end{cases}$$

One easily checks that $(u_j)_{j\in[\![1,N]\!]}$ is a solution of $\Sigma_\omega(\mathcal{G},L,\eta)$ for every $\eta\in\mathcal{D}$. Since it is periodic and nonzero, $\Sigma_\omega(\mathcal{G},L,\mathcal{D})$ is not exponentially stable in X_p^ω for any $p\in[1,+\infty]$, and so, by Corollary 4.64, $\Sigma_\omega(\mathcal{G},\Lambda,\mathcal{D})$ is not exponentially stable in X_p^ω for any $p\in[1,+\infty]$.

Suppose now that $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one vertex, and $\mathcal{D} = \overline{\mathcal{D}} \subset (0, +\infty)^d$. Then $\#\mathcal{V} = N + 1$ and, for every pair of points $q, p \in \mathcal{V}$, there exists a unique elementary path from q to p (see, for instance, [67]). Denote by q_0 the only vertex in $\mathcal{V}_{\mathrm{u}}$, set $\widehat{\mathcal{V}}_0 = \mathcal{V}_{\mathrm{u}}$, and, for $k \in \mathbb{N}^*$, let $\widehat{\mathcal{V}}_k$ be the set of vertices $q \in \mathcal{V}$ such that the unique elementary path from q to q_0 has length $k+1$. Let $K \in \mathbb{N}^*$ be the largest index for which $\widehat{\mathcal{V}}_K \neq \emptyset$; notice that $\widehat{\mathcal{V}}_k \neq \emptyset$ for every $k \in [\![0, K]\!]$ and that $\{\widehat{\mathcal{V}}_k\}_{k \in [\![0,K]\!]}$ forms a partition of $\mathcal{V}$. For $k \in [\![1, K]\!]$, let $\widehat{\mathcal{E}}_k = \{\{q, p\} \in \mathcal{E} \mid q \in \widehat{\mathcal{V}}_{k-1},\ p \in \widehat{\mathcal{V}}_k\}$; hence $\{\widehat{\mathcal{E}}_k\}_{k \in [\![1,K]\!]}$ is a partition of $\mathcal{E}$. Up to changing the orientation of the graph, we suppose that, for every $k \in [\![1, K]\!]$ and $e \in \widehat{\mathcal{E}}_k$, we have $\alpha(e) \in \widehat{\mathcal{V}}_{k-1}$ and $\omega(e) \in \widehat{\mathcal{V}}_k$. See Figure 4.2 for an illustration of these notations.

For $q, p \in \mathcal{V}$, let $(q = q_1, q_2, \ldots, q_n = p)$ be the unique elementary path from q to p and, for $i \in [\![1, n-1]\!]$, let j_i be the index corresponding to the edge $\{q_i, q_{i+1}\}$. We set

$$\widehat{\Lambda}_q^p = \sum_{i=1}^{n-1} \Lambda_{j_i} \qquad \text{and} \qquad \widehat{\Lambda} = \max_{q \in \mathcal{V}} \widehat{\Lambda}_q^{q_0}.$$

For $j \in \mathcal{E}$, let $\mathcal{Q}_j$ be the set of $q \in \mathcal{V}$ such that j is an edge in the unique elementary path from q to q_0.

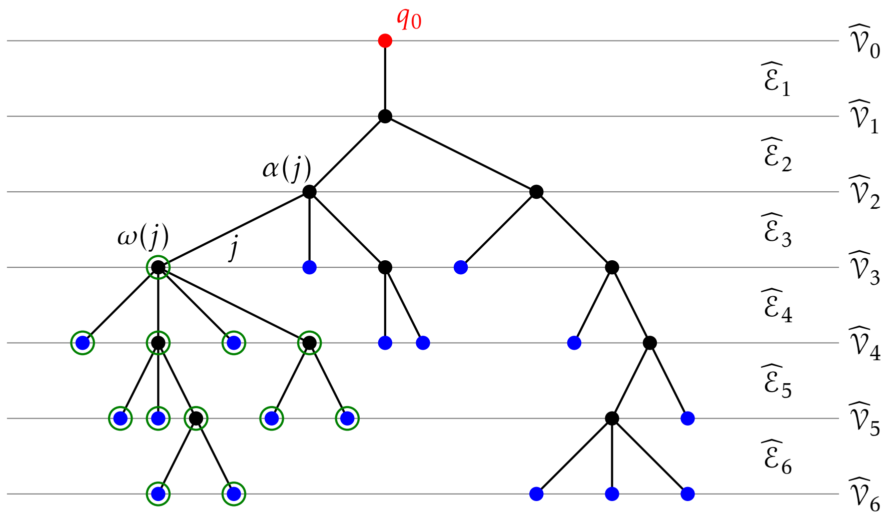


Figure 4.2: A tree $\mathcal{G}$ with $N = 28$ illustrating the notations used in this proof. Vertices in $\mathcal{V}_{\mathrm{d}}$ are marked in blue, the one in $\mathcal{V}_{\mathrm{u}}$ is marked in red and those in $\mathcal{V}_{\mathrm{int}}$ are marked in black. For the edge j represented in the figure, green circles were put around the vertices in $\mathcal{Q}_j$.

Let $\eta_{\min} = \min_{\eta \in \mathcal{D}} \min_{q \in \mathcal{V}_{\mathrm{d}}} \eta_q > 0$ and $\eta_{\max} = \max_{\eta \in \mathcal{D}} \max_{q \in \mathcal{V}_{\mathrm{d}}} \eta_q > 0$. Let $U = (u, v)$ be a solution of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ in X_2^ω and $f = \frac{1}{\sqrt{2}} T U$. Notice that $\|f(t)\|_{\mathsf{Y}_2(R)} = \|U(t)\|_2$ thanks to Remark 4.53. For $t \geq 0$, denote $F_0(t) = (f_i(t, 0))_{i \in [\![1, 2N]\!]}$ and $F_\Lambda(t) = (f_i(t, \Lambda_i^\tau))_{i \in [\![1, 2N]\!]}$, so that $F_0(t) = M(t) F_\Lambda(t)$. For $t \geq 0$ and $s \in [0, \Lambda_{\min}]$, we have, by Lemma 4.59,

$$\begin{aligned}
\|U(t+s)\|_2^2 &= \sum_{i=1}^{2N} \int_0^{\Lambda_i^\tau} |f_i(t+s, x)|^2 \, dx = \sum_{i=1}^{2N} \int_s^{\Lambda_i^\tau} |f_i(t, x-s)|^2 \, dx + \int_0^s |F_0(t+s-x)|_2^2 \, dx \\
&= \sum_{i=1}^{2N} \int_s^{\Lambda_i^\tau} |f_i(t, x-s)|^2 \, dx + \int_0^s |F_\Lambda(t+s-x)|_2^2 \, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^s \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \frac{4\eta_q(t+s-x)}{(1+\eta_q(t+s-x))^2} \lvert f_{2i}(t+s-x, \Lambda_i)\rvert^2 \, dx \\
&= \lVert U(t)\rVert_2^2 - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} \frac{4\eta_q(\tau)}{(1+\eta_q(\tau))^2} \lvert f_{2i}(\tau, \Lambda_i)\rvert^2 \, d\tau,
\end{aligned}
$$

and, since this holds for every $t \geq 0$ and every $s \in [0, \Lambda_{\min}]$, one can easily obtain by an inductive argument that, for every $t \geq 0$ and $s \geq 0$,

$$
\lVert U(t+s)\rVert_2^2 - \lVert U(t)\rVert_2^2 = - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} \frac{4\eta_q(\tau)}{(1+\eta_q(\tau))^2} \lvert f_{2i}(\tau, \Lambda_i)\rvert^2 \, d\tau.
$$

Thus, for every $t \geq 0$ and $s \geq 0$,

$$
\lVert U(t+s)\rVert_2^2 - \lVert U(t)\rVert_2^2 \leq - \frac{4\eta_{\min}}{(1+\eta_{\max})^2} \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} \lvert f_{2i}(\tau, \Lambda_i)\rvert^2 \, d\tau. \tag{4.58}
$$

In particular, $t \mapsto \lVert U(t)\rVert_2^2$ is nonincreasing.

Let $j \in \mathcal{E}$. If $\omega(j) \in \mathcal{V}_{\mathrm{d}}$, then, by combining Lemmas 4.66 and 4.68, we obtain that

$$
E_j(t) \leq 2 \int_{t-\Lambda_j}^{t+\Lambda_j} \left\lvert f_{2j}(\tau, \Lambda_j)\right\rvert^2 \, d\tau. \tag{4.59}
$$

Otherwise, we have $\omega(j) \in \mathcal{V}_{\mathrm{int}}$. Let $k \in [\![1, K]\!]$ be such that $j \in \widehat{\mathcal{E}}_k$ and let $q = \omega(j) \in \widehat{\mathcal{V}}_k$. Clearly, $k < K$ since $\widehat{\mathcal{V}}_K \subset \mathcal{V}_{\mathrm{d}}$. We claim that, for every $\ell \in [\![k, K]\!]$ and $t \geq \max_{i \in \mathcal{Q}_j} \widehat{\Lambda}_{\omega(i)}^{\alpha(j)}$, we have

$$
\begin{aligned}
E_j(t) \leq{}& (4N)^{\ell-k} \left[\sum_{\substack{i \in \widehat{\mathcal{E}}_\ell \\ \omega(i) \in \mathcal{Q}_j}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \left(\lvert f_{2i-1}(\tau, 0)\rvert^2 + \lvert f_{2i}(\tau, \Lambda_i)\rvert^2\right) d\tau \right. \\
& \left. + 2 \sum_{r=k+1}^{\ell-1} \sum_{\substack{i \in \widehat{\mathcal{E}}_r \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{d}}}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \lvert f_{2i}(\tau, \Lambda_i)\rvert^2 \, d\tau \right].
\end{aligned} \tag{4.60}
$$

Let us show (4.60) by induction on $\ell \in [\![k, K]\!]$. For $\ell = k$, (4.60) reads

$$
E_j(t) \leq \int_{t-\Lambda_j}^{t+\Lambda_j} \left(\left\lvert f_{2j-1}(\tau, 0)\right\rvert^2 + \left\lvert f_{2j}(\tau, \Lambda_j)\right\rvert^2\right) d\tau,
$$

which is exactly (4.57). Suppose now that $\ell \in [\![k, K-1]\!]$ is such that (4.60) holds and let $i \in \widehat{\mathcal{E}}_\ell$ be such that $\omega(i) \in \mathcal{Q}_j$. If $\omega(i) \in \mathcal{V}_{\mathrm{d}}$, then, by Lemma 4.68,

$$
\int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \left(\lvert f_{2i-1}(\tau, 0)\rvert^2 + \lvert f_{2i}(\tau, \Lambda_i)\rvert^2\right) d\tau \leq 2 \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \lvert f_{2i}(\tau, \Lambda_i)\rvert^2 \, d\tau. \tag{4.61}
$$

Otherwise, $\omega(i) \in \mathcal{V}_{\mathrm{int}}$, and then, by Lemma 4.67,

$$
\int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \left(\lvert f_{2i-1}(\tau, 0)\rvert^2 + \lvert f_{2i}(\tau, \Lambda_i)\rvert^2\right) d\tau
$$

$$\leq 4N \sum_{s \in \mathcal{E}_{\omega(i)} \setminus \{i\}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}-\Lambda_s}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}+\Lambda_s} \left(|f_{2s-1}(\tau,0)|^2 + |f_{2s}(\tau,\Lambda_s)|^2\right) d\tau$$

$$= 4N \sum_{s \in \mathcal{E}_{\omega(i)} \setminus \{i\}} \int_{t-\widehat{\Lambda}_{\omega(s)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(s)}^{\alpha(j)}} \left(|f_{2s-1}(\tau,0)|^2 + |f_{2s}(\tau,\Lambda_s)|^2\right) d\tau. \quad (4.62)$$

Combining (4.60), (4.61), and (4.62) gives

$$\begin{aligned} E_j(t) \leq (4N)^{\ell-k} \Bigg[& 4N \sum_{\substack{i \in \widehat{\mathcal{E}}_\ell \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{int}}}} \sum_{s \in \mathcal{E}_{\omega(i)} \setminus \{i\}} \int_{t-\widehat{\Lambda}_{\omega(s)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(s)}^{\alpha(j)}} \left(|f_{2s-1}(\tau,0)|^2 + |f_{2s}(\tau,\Lambda_s)|^2\right) d\tau \\ & + 2 \sum_{\substack{i \in \widehat{\mathcal{E}}_\ell \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{d}}}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau + 2 \sum_{r=k+1}^{\ell-1} \sum_{\substack{i \in \widehat{\mathcal{E}}_r \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{d}}}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau \Bigg] \\ \leq (4N)^{\ell+1-k} \Bigg[& \sum_{\substack{i \in \widehat{\mathcal{E}}_{\ell+1} \\ \omega(i) \in \mathcal{Q}_j}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} \left(|f_{2i-1}(\tau,0)|^2 + |f_{2i}(\tau,\Lambda_i)|^2\right) d\tau \\ & + 2 \sum_{r=k+1}^{\ell} \sum_{\substack{i \in \widehat{\mathcal{E}}_r \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{d}}}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau \Bigg], \end{aligned}$$

which establishes (4.60) for every $\ell \in [\![k, K]\!]$ by induction.

Applying (4.60) for $\ell = K$ and using the fact that $\widehat{\mathcal{V}}_K \subset \mathcal{V}_{\mathrm{d}}$ and Lemma 4.68, we obtain that, for $t \geq \widehat{\Lambda}$,

$$\begin{aligned} E_j(t) &\leq 2(4N)^{K-k} \sum_{r=k+1}^{K} \sum_{\substack{i \in \widehat{\mathcal{E}}_r \\ \omega(i) \in \mathcal{Q}_j \cap \mathcal{V}_{\mathrm{d}}}} \int_{t-\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}}^{t+\widehat{\Lambda}_{\omega(i)}^{\alpha(j)}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau \\ &\leq 2(4N)^K \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_{t-\widehat{\Lambda}}^{t+\widehat{\Lambda}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau. \end{aligned} \quad (4.63)$$

Equation (4.63) was established for $j \in [\![1, N]\!]$ such that $\omega(j) \in \mathcal{V}_{\mathrm{int}}$, but it also holds when $\omega(j) \in \mathcal{V}_{\mathrm{d}}$ thanks to (4.59). Hence, summing (4.63) over $j \in [\![1, N]\!]$, we get, for $t \geq 0$,

$$\left\| U(t+\widehat{\Lambda}) \right\|_2^2 \leq 2^{2K+1} N^{K+1} \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+2\widehat{\Lambda}} |f_{2i}(\tau,\Lambda_i)|^2 \, d\tau. \quad (4.64)$$

We now combine (4.58) with (4.64) to obtain, using the fact that $t \mapsto \|U(t)\|_2^2$ is nonincreasing, that, for every $t \geq 0$,

$$\left\| U(t+2\widehat{\Lambda}) \right\|_2^2 - \|U(t)\|_2^2 \leq -C \left\| U(t+2\widehat{\Lambda}) \right\|_2^2$$

with $C = \frac{4\eta_{\min}}{(1+\eta_{\max})^2} 2^{-2K-1} N^{-K} > 0$. This yields the required exponential convergence in X_2^ω, and hence in X_p^ω for every $p \in [1, +\infty]$ thanks to Corollary 4.64. ■

Chapter 5

Controllability of linear difference equations

5.1 Introduction

This chapter characterizes the controllability of the difference equation

$$\Sigma(A, B, \Lambda): \qquad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + B u(t), \tag{5.1}$$

where $x(t) \in \mathbb{C}^d$ is the state, $u(t) \in \mathbb{C}^m$ is the control input, $N, d, m \in \mathbb{N}^*$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$ is the vector of positive delays, $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$ is a N-tuple of $d \times d$ complex-valued matrices, and $B \in \mathcal{M}_{d,m}(\mathbb{C})$ is a $d \times m$ complex-valued matrix.

As presented in Section 1.4.1, the study of the autonomous difference equation

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) \tag{5.2}$$

has a long history and its analysis through spectral methods has led to important stability criteria, such as those in Theorems 1.36 and 1.39 (see also [14,60,64,84,94,129], [86, Chapter 9], and references therein).

A major motivation for analyzing the stability of (5.2) is that it is deeply related to properties of more general neutral functional differential equations of the form

$$\frac{d}{dt}\left(x(t) - \sum_{j=1}^{N} A_j x(t - \Lambda_j)\right) = f(x_t)$$

where $x_t : [-r, 0] \to \mathbb{C}^d$ is given by $x_t(s) = x(t+s)$, $r \geq \max_{j \in \{1, \dots, N\}} \Lambda_j$, and f is some function defined on a certain space (typically $\mathcal{C}^k([-r, 0], \mathbb{C}^d)$ or $W^{k,p}((-r, 0), \mathbb{C}^d)$); see Section 1.4.2 and also [60, 64, 84, 136], [86, Section 9.7]. Another important motivation is that, using d'Alembert decomposition and classical transformations of hyperbolic PDEs into differential or difference equations with delays based mainly on the method of characteristics, some hyperbolic PDEs can be put under the form (5.2) [48,54,70,106,160]. In particular, this has been done in the previous chapter in order to obtain stability results for transport and wave equations on networks with time-varying parameters from corresponding stability results for (5.2) with time-varying matrices A_j.

Several works in the literature have addressed the questions of control and stabilization of neutral functional differential equations [87, 140, 141, 143, 154] (see also Section 1.4.4), including for controlled difference equations of the form (5.1), such as the stabilization result from Theorem 1.43.

Concerning the controllability problem, due to the infinite-dimensional nature of the dynamics of difference equations and neutral functional differential equations, several different notions of controllability can be used, such as exact, approximate, spectral, or relative controllability [51, 154]. Relative controllability has been originally introduced in the study of control systems with delays in the control input [51, 105, 142], but this notion has later been extended and used to study also systems with delays in the state [66, 148] and in more general frameworks, such as for stochastic control systems [103] or fractional integro-differential systems [18]. The main idea of relative controllability is that, instead of controlling the state $x_t : [-r, 0] \to \mathbb{C}^d$ of (5.1), defined by $x_t(s) = x(t+s)$, in a certain function space such as $\mathcal{C}^k([-r, 0], \mathbb{C}^d)$ or $L^p((-r, 0), \mathbb{C}^d)$, where $r \geq \max_{j \in \{1, \dots, N\}} \Lambda_j$, one controls only the final state $x(t) = x_t(0)$. We defer the precise definition of relative controllability used in this chapter to Definition 5.15, after having proved in Theorems 5.12 and 5.13 criteria for several equivalent or closely related notions of relative controllability.

Despite the long history of the study of relative controllability, up to the author's knowledge, no general criterion allowing to characterize the relative controllability of (5.1) is available in the literature. The goal of this chapter is to fill this gap by providing necessary and sufficient conditions for the relative controllability of (5.1) in some different function spaces. We also discuss the dependence of such controllability on the delays $\Lambda_1, \dots, \Lambda_N$, and, more precisely, on their rational dependence structure, and provide an upper bound on the minimal time for controllability in terms of the dimension d of the system and its largest delay. Notice that some of these questions have already been addressed for particular systems under the form (5.1) in the literature, such as in Theorem 1.45 (see, e.g., [66, 148]). The main results of this chapter generalize those of these works.

We also consider in this chapter the exact and approximate controllability of (5.1) in the function space $L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$. Such problem is largely absent from the literature, with the notable exception of [154] and references therein, where duality arguments are used in order to characterize some controllability notions in terms of corresponding observability properties. The main results of this chapter concerning exact and approximate controllability are algebraic characterizations of such properties, first for commensurable delays, and then without the commensurability hypothesis for two-dimensional systems with two delays.

The main tool used in the analysis of the controllability of (5.1) in this chapter is a suitable representation formula for its solutions, describing a solution in time t in terms of its initial condition, the control input, and some matrix-valued coefficients computed recursively (see Proposition 5.8). Such formula generalizes the ones from Theorems 3.15 and 3.18, used in Chapter 3 to analyze the stability of a system of transport equations on a network under intermittent damping, and the one from Proposition 4.14, used in Chapter 4 to obtain stability criteria for (4.1), providing in particular a generalized version of the Hale–Silkowski stability criterion.

The plan of the chapter is as follows. After some general discussion on the well-posedness of (5.1) and the derivation of the explicit representation formula for its solutions in Section 5.2, we characterize relative controllability for some fixed final time $T > 0$ in Section 5.3.1 in the set of all functions and in the function spaces L^p and $\mathcal{C}^k$. For given $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $B \in \mathcal{M}_{d,m}(\mathbb{C})$, Section 5.3.2 compares the relative controllability of (5.1) for different delays $\Lambda_1, \dots, \Lambda_N$ and $L_1, \dots, L_N$ in terms of their rational dependence structure. Section 5.3.3 provides a uniform upper bound on the minimal time for the relative controllability of (5.1), an alternative proof of such result being provided in Appendix 5.A. The exact and approximate controllability of (5.1) in L^2 are the subject of Section 5.4, where we treat first the case of commensurable delays in Section 5.4.1, before characterizing exact and approximate controllability of (5.1) in dimension 2 with two delays under no commensurability assumptions in Section 5.4.2.

Notice that all the results in this chapter also hold, with the same proofs, if one assumes $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{R})^N$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ with the state $x(t) \in \mathbb{R}^d$ and the control $u(t) \in \mathbb{R}^m$. We choose complex-valued matrices, states, and controls for (5.1) in this chapter following the approach of Chapter 4, which is mainly motivated by the fact that classical spectral conditions for difference equations such as those from Theorems 1.36, 1.39, 1.41, 1.43, or 1.44 are more naturally written down in such framework.

5.2 Well-posedness and explicit representation of solutions

This sections establishes the well-posedness of (5.1) and provides an explicit representation formula for its solutions. The proofs of the main results of this section, Propositions 5.2 and 5.8, are very similar to the ones from Proposition 4.2 and Lemma 4.13 for the corresponding uncontrolled system. We start by providing the definition of solution used in this chapter.

Definition 5.1. Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $u : [0, T] \to \mathbb{C}^m$. We say that $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ is a *solution* of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u if it satisfies (5.1) for every $t \in [0, T]$ and $x(t) = x_0(t)$ for $t \in [-\Lambda_{\max}, 0)$. In this case, we set, for $t \in [0, T]$, $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0)}$.

Notice that, similarly to Definition 4.1, this notion of solution contains no regularity assumptions on x_0, u, or x. Nonetheless, such weak framework is enough to guarantee existence and uniqueness of solutions.

Proposition 5.2. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $u : [0, T] \to \mathbb{C}^m$. Then $\Sigma(A, B, \Lambda)$ admits a unique solution $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ with initial condition x_0 and control u.*

The proof of Proposition 5.2 is very similar to that of Proposition 4.2. We provide it here for the sake of completeness.

Proof. Let $T^* > 0$ be such that $T^* \leq T$ and $T^* < \Lambda_{\min}$. It suffices to build the solution x on $[-\Lambda_{\max}, T^*]$ and then complete its construction on $(T^*, T]$ by a standard inductive argument.

Suppose that $x : [-\Lambda_{\max}, T^*] \to \mathbb{C}^d$ is a solution of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u. Then

$$x(t) = \begin{cases} \displaystyle\sum_{j=1}^N A_j x_0(t - \Lambda_j) + B u(t), & \text{if } 0 \leq t \leq T^*, \\ x_0(t), & \text{if } -\Lambda_{\max} \leq t < 0. \end{cases} \tag{5.3}$$

Since the right-hand side is uniquely determined by x_0, u, A, and B, we obtain the uniqueness of the solution. Conversely, if $x : [-\Lambda_{\max}, T^*] \to \mathbb{C}^d$ is defined by (5.3), then (5.1) clearly holds for $t \in [0, T^*]$ and thus x is a solution of $\Sigma(A, B, \Lambda)$. ■

Remark 5.3. Let $T > 0$. If $x_0, \widetilde{x}_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and $u, \widetilde{u} : [0, T] \to \mathbb{C}^m$ are such that $x_0 = \widetilde{x}_0$ and $u = \widetilde{u}$ almost everywhere on their respective domains, it follows from (5.3) that the solutions $x, \widetilde{x} : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ of $\Sigma(A, B, \Lambda)$ associated respectively with x_0, u, and $\widetilde{x}_0$, $\widetilde{u}$, satisfy $x = \widetilde{x}$ almost everywhere on $[-\Lambda_{\max}, T]$. In particular, one still obtains existence and uniqueness of solutions of $\Sigma(A, B, \Lambda)$ (in the sense of functions defined almost everywhere) for initial conditions in $L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ and controls in $L^p((0, T), \mathbb{C}^m)$ for some $p \in [1, +\infty]$. Moreover, it follows easily from (5.3) that, in this case, solutions x of $\Sigma(A, B, \Lambda)$ satisfy $x \in L^p((-\Lambda_{\max}, T), \mathbb{C}^d)$, and hence $x_t \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ for every $t \in [0, T]$.

Remark 5.4. If $x_0 \in \mathcal{C}^k([-\Lambda_{\max}, 0), \mathbb{C}^d)$ and $u \in \mathcal{C}^k([0, T], \mathbb{C}^m)$ for some $k \in \mathbb{N}$, it follows from (5.3) that the corresponding solution x of $\Sigma(A, B, \Lambda)$ belongs to $\mathcal{C}^k([-\Lambda_{\max}, T], \mathbb{C}^d)$ if and only if x_0 and u satisfy the compatibility condition

$$\lim_{t \to 0} x_0^{(r)}(t) = \sum_{j=1}^N A_j x_0^{(r)}(-\Lambda_j) + B u^{(r)}(0), \qquad \forall r \in [\![0, k]\!], \tag{5.4}$$

where $x_0^{(r)}$ and $u^{(r)}$ denote the r-th derivatives of x_0 and u, respectively.

Due to the compatibility condition (5.4) required for obtaining solutions x in the space $\mathcal{C}^k([-\Lambda_{\max}, T], \mathbb{C}^d)$, we find it useful to introduce the following definition.

Definition 5.5. Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $k \in \mathbb{N}$. We say that x_0 is *$\mathcal{C}^k$-admissible* for system $\Sigma(A, B, \Lambda)$ if $x_0 \in \mathcal{C}^k([-\Lambda_{\max}, 0), \mathbb{C}^d)$ and, for every $r \in [\![0, k]\!]$, $\lim_{t \to 0} x_0^{(r)}(t)$ exists and

$$\lim_{t \to 0} x_0^{(r)}(t) - \sum_{j=1}^N A_j x_0^{(r)}(-\Lambda_j) \in \operatorname{Ran} B.$$

In order to provide an explicit representation for the solutions of $\Sigma(A, B, \Lambda)$, we first provide a recursive definition of the matrix coefficients $\Xi_{\mathbf{n}}$ appearing in such representation.

Definition 5.6. For $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $\mathbf{n} \in \mathbb{Z}^N$, we define the matrix $\Xi_{\mathbf{n}} \in \mathcal{M}_d(\mathbb{C})$ inductively by

$$\Xi_{\mathbf{n}} = \begin{cases} 0, & \text{if } \mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ \mathrm{Id}_d, & \text{if } \mathbf{n} = 0, \\ \displaystyle\sum_{k=1}^N A_k \Xi_{\mathbf{n} - e_k}, & \text{if } \mathbf{n} \in \mathbb{N}^N \setminus \{0\}. \end{cases} \tag{5.5}$$

Notice that this is the same definition as (4.5), but the matrix coefficients $\Xi_{\mathbf{n}}$ from (5.5) do not depend on the time t nor on the delay vector $\Lambda = (\Lambda_1, \dots, \Lambda_N)$, since we assume in this chapter that $A_1, \dots, A_N$ are constant. We also omit from the notation the dependence of $\Xi_{\mathbf{n}}$ on $A = (A_1, \dots, A_N)$.

Remark 5.7. It follows from Proposition 4.8 that, for $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N \setminus \{0\}$, the matrices $\Xi_{\mathbf{n}}$ also satisfy the recurrence relation

$$\Xi_{\mathbf{n}} = \sum_{k=1}^N \Xi_{\mathbf{n} - e_k} A_k$$

and they can be explicitly computed from $A = (A_1, \dots, A_N)$ by

$$\Xi_{\mathbf{n}} = \sum_{v \in V_{\mathbf{n}}} A_{v_1} A_{v_2} \cdots A_{v_{|\mathbf{n}|_1}},$$

where $V_{\mathbf{n}}$ is defined in (4.4) and can also be described by $V_{\mathbf{n}} = \{v \in [\![1, N]\!]^{|\mathbf{n}|_1} \mid \text{for every } k \in [\![1, N]\!],\ \#\{j \in [\![1, |\mathbf{n}|_1]\!] \mid v_j = k\} = n_k\}$.

We now provide an explicit representation for the solutions of $\Sigma(A, B, \Lambda)$, which is a generalization of Lemma 4.13 to the case of the controlled difference equation (5.1).

Proposition 5.8. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $u : [0, T] \to \mathbb{C}^m$. The corresponding solution $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ of $\Sigma(A, B, \Lambda)$ is given for $t \in [0, T]$ by*

$$x(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t}} \Xi_{\mathbf{n}} B u(t - \Lambda \cdot \mathbf{n}). \tag{5.6}$$

Proof. By linearity, it suffices to show that the function x_1 defined by

$$x_1(t) = \begin{cases} \displaystyle\sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}), & \text{if } 0 \leq t \leq T, \\ x_0(t), & \text{if } -\Lambda_{\max} \leq t < 0, \end{cases}$$

is the solution of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control 0, and that the function x_2 defined by

$$x_2(t) = \begin{cases} \displaystyle\sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t}} \Xi_{\mathbf{n}} B u(t - \Lambda \cdot \mathbf{n}), & \text{if } 0 \leq t \leq T, \\ 0, & \text{if } -\Lambda_{\max} \leq t < 0, \end{cases} \tag{5.7}$$

is the solution of $\Sigma(A, B, \Lambda)$ with initial condition 0 and control u. The first part has already been shown in Lemma 4.13, we are thus left to show that x_2 satisfies (5.1) for $t \in [0, T]$.

Let $j \in [\![1, N]\!]$. For $t \in [0, T]$, we have

$$x_2(t - \Lambda_j) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t - \Lambda_j}} \Xi_{\mathbf{n}} B u(t - \Lambda_j - \Lambda \cdot \mathbf{n}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{m} \leq t,\, m_j \geq 1}} \Xi_{\mathbf{m} - e_j} B u(t - \Lambda \cdot \mathbf{m}),$$

where we extend u by zero outside the interval $[0, T]$. Hence, using (5.5), we obtain that

$$\begin{aligned} \sum_{j=1}^N A_j x_2(t - \Lambda_j) &= \sum_{j=1}^N A_j \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{m} \leq t,\, m_j \geq 1}} \Xi_{\mathbf{m} - e_j} B u(t - \Lambda \cdot \mathbf{m}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{m} \leq t}} \sum_{\substack{j=1 \\ m_j \geq 1}}^N A_j \Xi_{\mathbf{m} - e_j} B u(t - \Lambda \cdot \mathbf{m}) \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{N}^N \setminus \{0\} \\ \Lambda \cdot \mathbf{m} \leq t}} \Xi_{\mathbf{m}} B u(t - \Lambda \cdot \mathbf{m}) = x_2(t) - B u(t), \end{aligned}$$

which shows that x_2 satisfies (5.1). ∎

Remark 5.9. When $A = (A_1, \dots, A_N)$ and B are time-varying, i.e. $A : [0, T] \to \mathcal{M}_d(\mathbb{C})^N$ and $B : [0, T] \to \mathcal{M}_{d,m}(\mathbb{C})$, the counterpart of Proposition 5.2 also holds with the same proof, and Remark 5.3 also applies, in the sense that solutions corresponding to A, B and $\widetilde{A}, \widetilde{B}$ are equal almost everywhere if $A = \widetilde{A}$ and $B = \widetilde{B}$ almost everywhere. The conclusion that $x \in L^p((-\Lambda_{\max}, T), \mathbb{C}^d)$ when $x_0 \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ and $u \in L^p((0, T), \mathbb{C}^m)$ holds under the extra assumption that $A \in L^\infty((0, T), \mathcal{M}_d(\mathbb{C})^N)$ and $B \in L^\infty((0, T), \mathcal{M}_{d,m}(\mathbb{C}))$. Moreover, the explicit formula from Proposition 5.8 becomes

$$x(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi^{\Lambda}_{\mathbf{n} - e_j, t} A_j(t - \Lambda \cdot \mathbf{n} + \Lambda_j) x_0(t - \Lambda \cdot \mathbf{n}) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t}} \Xi^{\Lambda}_{\mathbf{n}, t} B(t - \Lambda \cdot \mathbf{n}) u(t - \Lambda \cdot \mathbf{n}), \tag{5.8}$$

where the matrix coefficients $\Xi_{\mathbf{n},t}^{\Lambda}$ are defined in (4.5).

Remark 5.10. Let $p \in [1,+\infty]$. For $t \geq 0$, we define the bounded linear operator $S(t) : L^p((-\Lambda_{\max}, 0), \mathbb{C}^d) \to L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ by

$$(S(t)x_0)(s) = \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j\leq t+s-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j} A_j x_0(t+s-\Lambda\cdot\mathbf{n}).$$

The operator $S(t)$ maps an initial condition x_0 to the state $x_t = x(t+\cdot)|_{(-\Lambda_{\max},0)}$, where x is the solution of $\Sigma(A,B,\Lambda)$ at time t with initial condition x_0 and control 0. For $p \in [1,+\infty)$, the family $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup in $L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ (see Proposition 4.5).

The controllability results we establish in Section 5.3.1 are based on the explicit representation for the solutions from Proposition 5.8. Notice that the control u only affects the second term of (5.6). Since, in this term, u is evaluated only at times $t-\Lambda\cdot\mathbf{n}$, one should pack together coefficients $\Xi_{\mathbf{n}}$ corresponding to different $\mathbf{n}, \mathbf{n}' \in \mathbb{N}$ for which $\Lambda\cdot\mathbf{n} = \Lambda\cdot\mathbf{n}'$, in the same manner as in Definition 4.10.

Definition 5.11. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0,+\infty)^N$. We partition $\mathbb{N}^N$ according to the equivalence relation $\sim$ defined by writing $\mathbf{n} \sim \mathbf{n}'$ if $\Lambda\cdot\mathbf{n} = \Lambda\cdot\mathbf{n}'$. We use $[\cdot]_\Lambda$ to denote the equivalence classes of $\sim$ and we set $\mathcal{N}_\Lambda = \mathbb{N}^N/\sim$. The index Λ is omitted from the notation of $[\cdot]_\Lambda$ when the delay vector Λ is clear from the context. We define

$$\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} = \sum_{\mathbf{n}'\in[\mathbf{n}]} \Xi_{\mathbf{n}'}. \tag{5.9}$$

Thanks to Definition 5.11, the representation formula (5.6) for the solutions of $\Sigma(A,B,\Lambda)$ can be written as

$$x(t) = \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j\leq t-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j} A_j x_0(t-\Lambda\cdot\mathbf{n}) + \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda \\ \Lambda\cdot\mathbf{n}\leq t}} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bu(t-\Lambda\cdot\mathbf{n}). \tag{5.10}$$

5.3 Relative controllability

5.3.1 Relative controllability criteria

This section presents the main relative controllability criteria from the chapter, Theorems 5.12 and 5.13 below. Theorem 5.12 provides a criterion for relative controllability in the set of all functions and in the L^p spaces, whereas the criterion in Theorem 5.13 characterizes relative controllability in the $\mathcal{C}^k$ spaces. Both algebraic criteria we obtain are expressed in terms of the coefficients $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda}$ and the matrix B and are generalizations of the usual Kalman condition for the controllability of a discrete-time system. Their proofs are based on the explicit representation for solutions (5.10).

Theorem 5.12. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0,+\infty)^N$, $T > 0$, and $p \in [1,+\infty]$. Define $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda}$ as in (5.9). Then the following assertions are equivalent.*

(a) *One has*

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bw \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n}\leq T,\ w\in\mathbb{C}^m\right\} = \mathbb{C}^d. \tag{5.11}$$

(b) *For every* $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ *and* $x_1 \in \mathbb{C}^d$, *there exists* $u : [0, T] \to \mathbb{C}^m$ *such that the solution* x *of* $\Sigma(A, B, \Lambda)$ *with initial condition* x_0 *and control* u *satisfies* $x(T) = x_1$.

(c) *There exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, *and* $x_1 : [0, \varepsilon] \to \mathbb{C}^d$, *there exists* $u : [0, T + \varepsilon] \to \mathbb{C}^m$ *such that the solution* x *of* $\Sigma(A, B, \Lambda)$ *with initial condition* x_0 *and control* u *satisfies* $x(T + \cdot)|_{[0,\varepsilon]} = x_1$.

(d) *There exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$, *and* $x_1 \in L^p((0, \varepsilon), \mathbb{C}^d)$, *there exists* $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$ *such that the solution* x *of* $\Sigma(A, B, \Lambda)$ *with initial condition* x_0 *and control* u *satisfies* $x \in L^p((-\Lambda_{\max}, T + \varepsilon), \mathbb{C}^d)$ *and* $x(T + \cdot)|_{[0,\varepsilon]} = x_1$.

Proof. For $T > 0$, let $\mathcal{N}^T = \{[\mathbf{n}] \in \mathcal{N}_\Lambda \mid \Lambda \cdot \mathbf{n} \leq T\}$ and $n_T = \#\mathcal{N}^T$. The proof is carried out as follows. Clearly, (c) $\implies$ (b). We will show the equivalences by proving that (b) $\implies$ (a), (a) $\implies$ (c) and (d), and (d) $\implies$ (a).

Assume that (b) is satisfied, which shows, using (5.10) and considering a zero initial condition, that, for every $x_1 \in \mathbb{C}^d$, there exists $u : [0, T] \to \mathbb{C}^m$ such that

$$\left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} \left(u(T - \Lambda \cdot \mathbf{n})\right)_{[\mathbf{n}] \in \mathcal{N}^T} = \sum_{[\mathbf{n}] \in \mathcal{N}^T} \widehat{\Xi}^\Lambda_{[\mathbf{n}]} B u(T - \Lambda \cdot \mathbf{n}) = x_1, \tag{5.12}$$

where $\left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B\right)_{[\mathbf{n}] \in \mathcal{N}^T}$ denotes the $d \times m n_T$ matrix composed of the n_T blocks $\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B$ of size $d \times m$ and $\left(u(T - \Lambda \cdot \mathbf{n})\right)_{[\mathbf{n}] \in \mathcal{N}^T}$ denotes the $m n_T \times 1$ matrix composed of the n_T blocks $u(T - \Lambda \cdot \mathbf{n})$ of size $m \times 1$. This means that the map $\mathbb{C}^{m n_T} \ni U \mapsto \left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} U \in \mathbb{C}^d$ is surjective, and thus (a) is satisfied.

Assume now that (a) is satisfied and let

$$\varepsilon_0 = \min \left\{ \min_{\substack{[\mathbf{n}'], [\mathbf{n}] \in \mathcal{N}^T \\ [\mathbf{n}'] \neq [\mathbf{n}]}} \left|\Lambda \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}'\right|, \min_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} > T}} (\Lambda \cdot \mathbf{n} - T) \right\} > 0.$$

Let $\varepsilon \in (0, \varepsilon_0)$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, and $x_1 : [0, \varepsilon] \to \mathbb{C}^d$. Thanks to (a), the map $\mathbb{C}^{m n_T} \ni U \mapsto \left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} U \in \mathbb{C}^d$ is surjective, and hence the $d \times m n_T$ matrix $\left(\widehat{\Xi}^\Lambda_{[\mathbf{n}]} B\right)_{[\mathbf{n}] \in \mathcal{N}^T}$ admits a right inverse $M \in \mathcal{M}_{m n_T, d}(\mathbb{C})$. Let $U = \left(U_{[\mathbf{n}]}\right)_{[\mathbf{n}] \in \mathcal{N}^T} : [0, \varepsilon] \to \mathbb{C}^{m n_T} = (\mathbb{C}^m)^{\mathcal{N}^T}$ be given by

$$U(t) = M \left(x_1(t) - \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq T + t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(T + t - \Lambda \cdot \mathbf{n}) \right). \tag{5.13}$$

Define $u : [0, T + \varepsilon] \to \mathbb{C}^m$ by

$$u(t) = \begin{cases} U_{[\mathbf{n}]}(\Lambda \cdot \mathbf{n} + t - T), & \text{if } t \in [T - \Lambda \cdot \mathbf{n}, T - \Lambda \cdot \mathbf{n} + \varepsilon] \text{ for some } [\mathbf{n}] \in \mathcal{N}^T, \\ 0, & \text{otherwise}. \end{cases} \tag{5.14}$$

Thanks to the definition of ε_0, u is well-defined, and one has $u(T + t - \Lambda \cdot \mathbf{n}) = U_{[\mathbf{n}]}(t)$ for every $[\mathbf{n}] \in \mathcal{N}^T$ and $t \in [0, \varepsilon]$. Hence, it follows from (5.13) that, for every $t \in [0, \varepsilon]$,

$$x_1(t) - \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq T + t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(T + t - \Lambda \cdot \mathbf{n}) = \left(\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} \left(u(T + t - \Lambda \cdot \mathbf{n})\right)_{[\mathbf{n}] \in \mathcal{N}^T}$$
$$= \sum_{[\mathbf{n}] \in \mathcal{N}^T} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B u(T + t - \Lambda \cdot \mathbf{n}) = \sum_{\substack{[\mathbf{n}] \in \mathcal{N}_\Lambda \\ \Lambda \cdot \mathbf{n} \leq T + t}} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B u(T + t - \Lambda \cdot \mathbf{n}), \tag{5.15}$$

where we use that, thanks to the definition of ε_0, one has

$$\mathcal{N}^T = \{[\mathbf{n}] \in \mathcal{N}_\Lambda \mid \Lambda \cdot \mathbf{n} \leq T + t\}, \qquad \forall t \in [0, \varepsilon]. \tag{5.16}$$

It now follows from (5.10) and (5.15) that the solution x of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u satisfies $x(T + \cdot)|_{[0, \varepsilon]} = x_1$, and hence (c) holds. Notice moreover that, if we assume $x_0 \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$ and $x_1 \in L^p((0, \varepsilon), \mathbb{C}^d)$, it follows from (5.13) that $U \in L^p((0, \varepsilon), \mathbb{C}^{m n_T})$, and thus, by (5.14), $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$. Hence, the solution x of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u satisfies $x \in L^p((-\Lambda_{\max}, T + \varepsilon), \mathbb{C}^d)$, thanks to Remark 5.3, and $x(T + \cdot)|_{[0, \varepsilon]} = x_1$, which shows that (d) also holds.

Finally, assume that (d) holds, take $\varepsilon_0 > 0$ as in (d) and fix $\varepsilon \in (0, \varepsilon_0)$. Then, considering a zero initial condition, for every constant final state $x_1 \in \mathbb{C}^d$, there exists $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$ such that, for almost every $t \in (0, \varepsilon)$, one has, as in (5.12),

$$\left(\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} \left(u(T + t - \Lambda \cdot \mathbf{n})\right)_{[\mathbf{n}] \in \mathcal{N}^T} = x_1,$$

where we use that (5.16) holds, up to choosing a smaller $\varepsilon \in (0, \varepsilon_0)$. Hence, as in (5.12), one also obtains that the map $\mathbb{C}^{m n_T} \ni U \mapsto \left(\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B\right)_{[\mathbf{n}] \in \mathcal{N}^T} U \in \mathbb{C}^d$ is surjective, and thus (a) is satisfied. ∎

The next result presents a relative controllability criterion for $\mathcal{C}^k$ solutions of $\Sigma(A, B, \Lambda)$, which is slightly different from (a) in Theorem 5.12 due to the compatibility condition (5.4) required for the existence of $\mathcal{C}^k$ solutions.

Theorem 5.13. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, and $k \in \mathbb{N}$. Define $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda}$ as in (5.9). Then the following assertions are equivalent.*

(a) *One has*
$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} < T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d. \tag{5.17}$$

(b) *For every x_0 $\mathcal{C}^k$-admissible for $\Sigma(A, B, \Lambda)$ and $x_1 \in \mathbb{C}^d$, there exists $u \in \mathcal{C}^k([0, T], \mathbb{C}^m)$ such that the solution x of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u satisfies $x \in \mathcal{C}^k([-\Lambda_{\max}, T], \mathbb{C}^d)$ and $x(T) = x_1$.*

(c) *There exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, x_0 $\mathcal{C}^k$-admissible for $\Sigma(A, B, \Lambda)$, and $x_1 \in \mathcal{C}^k([0, \varepsilon], \mathbb{C}^d)$, there exists $u \in \mathcal{C}^k([0, T + \varepsilon], \mathbb{C}^m)$ such that the solution x of $\Sigma(A, B, \Lambda)$ with initial condition x_0 and control u satisfies $x \in \mathcal{C}^k([-\Lambda_{\max}, T + \varepsilon], \mathbb{C}^d)$ and $x(T + \cdot)|_{[0, \varepsilon]} = x_1$.*

Proof. Let $\mathcal{N}_*^T = \{[\mathbf{n}]_\Lambda \in \mathcal{N}_\Lambda \mid \Lambda \cdot \mathbf{n} < T\}$ and $n_T^* = \# \mathcal{N}_*^T$. We begin the proof by noticing that (c) implies (b). Assume now that (b) holds and let us show that (a) is satisfied. For every

$x_1 \in \mathbb{C}^d$, there exists $u \in \mathcal{C}^k([0,T],\mathbb{C}^m)$ such that the solution x of $\Sigma(A,B,\Lambda)$ with zero initial condition and control u satisfies $x \in \mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$ and, from (5.10),

$$\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\leq T}} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bu(T-\Lambda\cdot\mathbf{n}) = x_1. \tag{5.18}$$

Moreover, since $x \in \mathcal{C}^k([-\Lambda_{\max},T],\mathbb{C}^d)$, it follows from Remark 5.4 that (5.4) is satisfied, and thus, for every $r \in [\![0,k]\!]$, $Bu^{(r)}(0) = 0$. Thus (5.18) becomes

$$\sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}< T}} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bu(T-\Lambda\cdot\mathbf{n}) = x_1,$$

and we conclude, as in the proof of Theorem 5.12, that the map $\mathbb{C}^{mn_T^*} \ni U \mapsto \left(\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T} U \in \mathbb{C}^d$ is surjective, and thus (a) is satisfied.

Finally, assume that (a) is satisfied and let

$$\varepsilon_0 = \frac{1}{2}\min\left\{\min_{\substack{[\mathbf{n}'],[\mathbf{n}]\in\mathcal{N}_*^T\\ [\mathbf{n}']\neq[\mathbf{n}]}} \left|\Lambda\cdot\mathbf{n}-\Lambda\cdot\mathbf{n}'\right|, \min_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \Lambda\cdot\mathbf{n}\neq T}} \left|\Lambda\cdot\mathbf{n}-T\right|\right\} > 0.$$

Let $\varepsilon \in (0,\varepsilon_0)$, x_0 $\mathcal{C}^k$-admissible for $\Sigma(A,B,\Lambda)$, and $x_1 \in \mathcal{C}^k([0,\varepsilon],\mathbb{C}^d)$. Since x_0 is $\mathcal{C}^k$-admissible, there exists $\mu \in \mathcal{C}^k([0,\varepsilon],\mathbb{C}^m)$, with a compact support inside $[0,\varepsilon)$, such that, for every $r \in [\![0,k]\!]$,

$$\lim_{t\to 0} x_0^{(r)}(t) = \sum_{j=1}^N A_j x_0^{(r)}(-\Lambda_j) + B\mu^{(r)}(0). \tag{5.19}$$

If $T = \Lambda\cdot\mathbf{n}$ for some $\mathbf{n}\in\mathbb{N}^N$, we set $\delta_T = 1$ and $\tau = [\mathbf{n}]$; otherwise, we set $\delta_T = 0$ and $\tau = [0]$. As in the proof of Theorem 5.12, it follows from (a) that the $d\times mn_T^*$ matrix $\left(\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}$ admits a right inverse $M \in \mathcal{M}_{mn_T^*,d}(\mathbb{C})$. Let $U = \left(U_{[\mathbf{n}]}\right)_{[\mathbf{n}]\in\mathcal{N}_*^T} : [0,\varepsilon] \to \mathbb{C}^{mn_T^*} = (\mathbb{C}^m)^{\mathcal{N}_*^T}$ be given by

$$U(t) = M\left(x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\leq T+t-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j} A_j x_0(T+t-\Lambda\cdot\mathbf{n}) - \delta_T \widehat{\Xi}_\tau^\Lambda B\mu(t)\right). \tag{5.20}$$

Notice that the sum in (5.20) can be taken over the set

$$G_1(t) = \{(\mathbf{n} = (n_1,\dots,n_N), j) \in \mathbb{N}^N\times[\![1,N]\!] \mid -\Lambda_j \leq T+t-\Lambda\cdot\mathbf{n} < 0,\ n_j \geq 1\},$$

since $\Xi_{\mathbf{n}} = 0$ if $\mathbf{n}\in\mathbb{Z}^N\setminus\mathbb{N}^N$. Moreover, thanks to the definition of ε_0, one has $G_1(t) = G_1(0)$ for every $t\in[0,\varepsilon]$, and thus U can be written for $t\in[0,\varepsilon]$ as

$$U(t) = M\left(x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\leq T-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j} A_j x_0(T+t-\Lambda\cdot\mathbf{n}) - \delta_T \widehat{\Xi}_\tau^\Lambda B\mu(t)\right).$$

In particular, one obtains that $U \in \mathcal{C}^k([0,\varepsilon],\mathbb{C}^{mn_T^*})$. We extend U into a $\mathcal{C}^k$ function on the interval $\left[-\frac{\varepsilon}{2},\frac{3\varepsilon}{2}\right]$ with a compact support in $\left(-\frac{\varepsilon}{2},\frac{3\varepsilon}{2}\right)$. Define $u : [0,T+\varepsilon] \to \mathbb{C}^m$ by

$$u(t) = \begin{cases} U_{[\mathbf{n}]}(\Lambda\cdot\mathbf{n}+t-T), & \text{if } t \in \left[T-\Lambda\cdot\mathbf{n}-\frac{\varepsilon}{2}, T-\Lambda\cdot\mathbf{n}+\frac{3\varepsilon}{2}\right] \text{ for some } [\mathbf{n}] \in \mathcal{N}_*^T,\\ \mu(t), & \text{if } t \in [0,\varepsilon],\\ 0, & \text{otherwise,}\end{cases}$$

which is well-defined thanks to the choice of ε_0, and satisfies $u \in \mathcal{C}^k([0,T+\varepsilon],\mathbb{C}^m)$ thanks to the construction of U and μ. Moreover, one has $u(T+t-\Lambda\cdot\mathbf{n}) = U_{[\mathbf{n}]}(t)$ for every $[\mathbf{n}] \in \mathcal{N}_*^T$ and, thanks to (5.19), it follows from Remark 5.4 that the unique solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x \in \mathcal{C}^k([-\Lambda_{\max},T+\varepsilon],\mathbb{C}^d)$. It follows from (5.20) that, for every $t \in [0,\varepsilon]$,

$$\begin{aligned} &x_1(t) - \sum_{\substack{(\mathbf{n},j)\in\mathbb{N}^N\times[\![1,N]\!]\\ -\Lambda_j\leq T+t-\Lambda\cdot\mathbf{n}<0}} \Xi_{\mathbf{n}-e_j}A_j x_0(T+t-\Lambda\cdot\mathbf{n})\\ &= \delta_T\widehat{\Xi}_\tau^\Lambda B\mu(t) + \left(\widehat{\Xi}_{[\mathbf{n}]}^\Lambda B\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}\left(u(T+t-\Lambda\cdot\mathbf{n})\right)_{[\mathbf{n}]\in\mathcal{N}_*^T}\\ &= \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\leq T}} \widehat{\Xi}_{[\mathbf{n}]}^\Lambda Bu(T+t-\Lambda\cdot\mathbf{n}) = \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \Lambda\cdot\mathbf{n}\leq T+t}} \widehat{\Xi}_{[\mathbf{n}]}^\Lambda Bu(T+t-\Lambda\cdot\mathbf{n}),\end{aligned}$$

and hence the solution x of $\Sigma(A,B,\Lambda)$ with initial condition x_0 and control u satisfies $x(T+\cdot)|_{[0,\varepsilon]} = x_1$, which shows that (c) holds. ■

Remark 5.14. When $N = 1$, the controlled difference equation (5.1) becomes $x(t) = Ax(t-\Lambda) + Bu(t)$, with $A = A_1$ and $\Lambda = \Lambda_1$. It follows from Definitions 5.6 and 5.11 that, for $\mathbf{n} = n \in \mathbb{N}$, one has $\widehat{\Xi}_{[\mathbf{n}]}^\Lambda = A^n$, and thus condition (a) from Theorem 5.12 reduces to

$$\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda\rfloor}B\end{pmatrix} = d,$$

which is the usual Kalman condition for controllability of discrete-time linear systems (see, e.g., [163, Theorem 2]). Moreover, condition (a) from Theorem 5.13 reduces to

$$\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lceil T/\Lambda\rceil-1}B\end{pmatrix} = d,$$

which is the same as the previous one when $T/\Lambda \notin \mathbb{N}^*$.

Notice that (b), (c), and (d) from Theorem 5.12 and (b) and (c) from Theorem 5.13 could all be used to define relative controllability in different function spaces. Motivated by the equivalences established in Theorems 5.12 and 5.13, we provide the following definition.

Definition 5.15. Let $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda \in (0,+\infty)^N$, and $T > 0$.

(a) We say that $\Sigma(A,B,\Lambda)$ is *relatively controllable* in time T if

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^\Lambda Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d.$$

(b) If $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$, we define the *minimal controllability time* $T_{\min}$ for $\Sigma(A,B,\Lambda)$ by $T_{\min} = \inf\{T > 0 \mid \Sigma(A,B,\Lambda)$ is relatively controllable in time $T\}$.

Remark 5.16. Contrarily to the situation for linear control systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$ or $x(t) = Ax(t-1) + Bu(t)$, relative controllability for some time $T > 0$ does not imply stabilizability by a linear feedback law. Indeed, for $N = d = 2$ and $m = 1$, consider the system $\Sigma(A, B, \Lambda)$ with $A = (A_1, A_2)$, B, and $\Lambda = (\Lambda_1, \Lambda_2)$ given by

$$\begin{aligned} A_1 &= \begin{pmatrix} \alpha & -\alpha^{1-\ell} \\ 0 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \Lambda_1 &= 1, & \Lambda_2 &= \ell, \end{aligned}$$

with $\ell \in (0, 1)$ and $\alpha > 1$. Clearly, $\Sigma(A, B, \Lambda)$ is relatively controllable in time $T \geq \ell$ since $\operatorname{Span}\{B, A_2 B\} = \mathbb{C}^2$. However, for $\lambda \in \mathbb{C}$, one has

$$\operatorname{Id}_2 - A_1 e^{-\lambda} - A_2 e^{-\lambda\ell} = \begin{pmatrix} 1 - \alpha e^{-\lambda} & \alpha^{1-\ell} e^{-\lambda} - e^{-\lambda\ell} \\ 0 & 1 \end{pmatrix},$$

and the first row of this matrix is zero for $\lambda = \log\alpha$. Hence (1.60) does not hold for $\lambda = \log\alpha > 0$, and it follows from Theorem 1.43 that $\Sigma(A, B, \Lambda)$ cannot be strongly stabilized by a linear feedback law.

5.3.2 Rational dependence of the delays

This section compares relative controllability of $\Sigma(A, B, \Lambda)$ for different delay vectors Λ in terms of their rational dependence structure. We start by recalling the definition of rational dependence and commensurability.

Definition 5.17. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in \mathbb{R}^N$.

(a) We say that the components of Λ are *rationally dependent* if there exists $\mathbf{n} \in \mathbb{Z}^N \setminus \{0\}$ such that $\Lambda \cdot \mathbf{n} = 0$. Otherwise, the components of Λ are said to be *rationally independent*.

(b) We say that the components of Λ are *commensurable* if there exist $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}^N$ such that $\Lambda = \lambda k$.

Notice that the set $\mathbb{Z}^N$ can be replaced by $\mathbb{Q}^N$ in Definition 5.17 without changing the definitions of rational dependence and commensurability. We next introduce a preorder in the set of all possible delay vectors $(0, +\infty)^N$, which describes when one delay vector is "less rationally dependent" than another.

Definition 5.18. For $\Lambda \in (0, +\infty)^N$, we define $Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\}$. For $\Lambda, L \in (0, +\infty)^N$, we write $\Lambda \preccurlyeq L$ or, equivalently, $L \succcurlyeq \Lambda$, if $Z(\Lambda) \subset Z(L)$. We write $\Lambda \approx L$ if $\Lambda \preccurlyeq L$ and $L \preccurlyeq \Lambda$.

Notice that the sets $V_+(\Lambda)$ and $W_+(\Lambda)$ defined in (4.8) can be written, in terms of the preorder $\preccurlyeq$, as $V_+(\Lambda) = \{L \in (0, +\infty)^N \mid \Lambda \preccurlyeq L\}$ and $W_+(\Lambda) = \{L \in (0, +\infty)^N \mid \Lambda \approx L\}$. If $\Lambda \in (0, +\infty)^N$ has rationally independent components, then one immediately computes $Z(\Lambda) = \{0\}$, and hence $\Lambda \preccurlyeq L$ for every $L \in (0, +\infty)^N$, that is, delay vectors with rationally independent components are minimal for the preorder $\preccurlyeq$. Notice also that, for $\Lambda \in (0, +\infty)^N$, the set $Z(\Lambda)$ encodes the structure of the equivalence classes $[\mathbf{n}]_\Lambda$ for $\mathbf{n} \in \mathbb{N}^N$, in the sense that, for $\mathbf{n}' \in \mathbb{N}^N$, one has $\mathbf{n}' \in [\mathbf{n}]_\Lambda$ if and only if $\mathbf{n}' - \mathbf{n} \in Z(\Lambda)$, which shows that $[\mathbf{n}]_\Lambda = (\mathbf{n} + Z(\Lambda)) \cap \mathbb{N}^N$. The next proposition gathers some immediate properties that follow from Definition 5.18.

Proposition 5.19. *Let $\Lambda, L \in (0, +\infty)^N$. If $\Lambda \preccurlyeq L$, then, for every $\mathbf{n} \in \mathbb{N}^N$, one has $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ and*

$$\widehat{\Xi}^L_{[\mathbf{n}]} = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}^\Lambda_\tau. \tag{5.21}$$

In particular, if $\Lambda \approx L$, then, for every $\mathbf{n} \in \mathbb{N}^N$, one has $[\mathbf{n}]_\Lambda = [\mathbf{n}]_L$ and $\widehat{\Xi}^\Lambda_{[\mathbf{n}]} = \widehat{\Xi}^L_{[\mathbf{n}]}$.

Proof. If $\Lambda \preccurlyeq L$ and $\mathbf{n} \in \mathbb{N}^N$, the inclusion $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ follows immediately from the fact that $Z(\Lambda) \subset Z(L)$ and that $[\mathbf{n}]_\lambda = (\mathbf{n} + Z(\lambda)) \cap \mathbb{N}^N$ for every $\mathbf{n} \in \mathbb{N}^N$ and $\lambda \in (0, +\infty)^N$. Moreover, the set $\{\tau \in \mathcal{N}_\Lambda \mid \tau \subset [\mathbf{n}]_L\}$ is a partition of $[\mathbf{n}]_L$, since, for every $\mathbf{n}' \in [\mathbf{n}]_L$, one has $[\mathbf{n}']_\Lambda \subset [\mathbf{n}']_L = [\mathbf{n}]_L$ and all equivalence classes in $\mathcal{N}_\Lambda$ are disjoint. Hence

$$\sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}_\tau^\Lambda = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \sum_{\mathbf{n}' \in \tau} \Xi_{\mathbf{n}'} = \sum_{\mathbf{n}' \in [\mathbf{n}]_L} \Xi_{\mathbf{n}'} = \widehat{\Xi}_{[\mathbf{n}]}^L.$$

The statements in the case $\Lambda \approx L$ follow immediately. ∎

The first main result of this section is the following theorem.

Theorem 5.20. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda, L \in (0, +\infty)^N$, and $T > 0$ be such that $\Lambda \preccurlyeq L$. Set $\kappa = \max_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$. If $\Sigma(A, B, L)$ is relatively controllable in time T, then $\Sigma(A, B, \Lambda)$ is relatively controllable in time κT.*

Proof. Notice that, for every $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N \setminus \{0\}$, one has $\frac{\Lambda \cdot \mathbf{n}}{L \cdot \mathbf{n}} = \sum_{j=1}^N \frac{\Lambda_j}{L_j} \frac{L_j n_j}{L \cdot \mathbf{n}} \leq \kappa$, and thus $\Lambda \cdot \mathbf{n} \leq \kappa L \cdot \mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$. Using Proposition 5.19, one obtains that

$$\begin{aligned}
&\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^L B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_L,\, L \cdot \mathbf{n} \leq T,\, w \in \mathbb{C}^m\right\} \\
&= \operatorname{Span}\left\{\sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}_\tau^\Lambda B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_L,\, L \cdot \mathbf{n} \leq T,\, w \in \mathbb{C}^m\right\} \\
&\subset \operatorname{Span}\left\{\widehat{\Xi}_\tau^\Lambda B w \,\middle|\, \tau \in \mathcal{N}_\Lambda,\, \tau \subset [\mathbf{n}]_L,\, [\mathbf{n}]_L \in \mathcal{N}_L,\, L \cdot \mathbf{n} \leq T,\, w \in \mathbb{C}^m\right\} \\
&= \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^\Lambda B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\, L \cdot \mathbf{n} \leq T,\, w \in \mathbb{C}^m\right\} \\
&\subset \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^\Lambda B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\, \Lambda \cdot \mathbf{n} \leq \kappa T,\, w \in \mathbb{C}^m\right\},
\end{aligned}$$

which proves the statement. ∎

Theorem 5.20 proves that relative controllability of $\Sigma(A, B, L)$ implies that of $\Sigma(A, B, \Lambda)$ for all delay vectors Λ such that $\Lambda \preccurlyeq L$ (with different controllability times). The converse of this result does not hold, as illustrated in the following example.

Example 5.21. Consider the system $\Sigma(A, B, \Lambda)$ with $N = 2$, $d = 3$, $m = 1$, $\Lambda = (1, \lambda)$ for some $\lambda \in (0, 1)$, and

$$A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

One has $A_1 = -A_2^2$ and hence one immediately computes

$$\Xi_{\mathbf{n}} = \begin{cases} \mathrm{Id}_3, & \text{if } \mathbf{n} = (0,0), \\ A_1, & \text{if } \mathbf{n} = (1,0), \\ A_2, & \text{if } \mathbf{n} = (0,1), \\ A_2^2, & \text{if } \mathbf{n} = (0,2), \\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda \notin \mathbb{Q}$, one has $\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} = \Xi_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{N}^2$, and thus, for every $T \geq 1$,

$$\begin{aligned}\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \,\middle|\, [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}\right\} &= \operatorname{Span}\left\{\Xi_{\mathbf{n}} B \,\middle|\, \mathbf{n} = (n_1, n_2) \in \mathbb{N}^2,\ n_1 + \lambda n_2 \leq T\right\} \\ &\supset \operatorname{Span}\{\Xi_{(0,0)} B, \Xi_{(1,0)} B, \Xi_{(0,1)} B\} = \mathbb{C}^3,\end{aligned}$$

which shows that $\Sigma(A, B, \Lambda)$ is relatively controllable for every $T \geq 1$ when $\lambda \notin \mathbb{Q}$. However, for $\lambda = \frac{1}{2}$, one computes

$$\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} = \begin{cases} \mathrm{Id}_3, & \text{if } [\mathbf{n}] = [(0,0)], \\ A_2, & \text{if } [\mathbf{n}] = [(0,1)], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for every $T > 0$,

$$\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} Bw \,\middle|\, [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}\right\} \subset \operatorname{Span}\{B, A_2 B\} \subsetneq \mathbb{C}^3,$$

and hence $\Sigma(A, B, \Lambda)$ is not relatively controllable for any $T > 0$ when $\lambda = \frac{1}{2}$.

Even if the converse of Theorem 5.20 does not hold in general, one can still obtain that relative controllability with a delay vector $\Lambda \in (0, +\infty)^N$ implies relative controllability for another delay vector $L \succcurlyeq \Lambda$ with commensurable components and sufficiently close to Λ.

Theorem 5.22. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, and $T > 0$. For every $\varepsilon > 0$, there exists $L = (L_1, \dots, L_N) \in (0, +\infty)^N$ with commensurable components satisfying $L \succcurlyeq \Lambda$ and $1 \leq \frac{\Lambda_j}{L_j} < 1 + \varepsilon$ for every $j \in [\![1, N]\!]$ such that, if $\Sigma(A, B, \Lambda)$ is relatively controllable in time T, then $\Sigma(A, B, L)$ is also relatively controllable in time T.*

Before proving Theorem 5.22, let us show the following result.

Lemma 5.23. *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$ and $T > 0$. For every $\varepsilon > 0$, there exists $L = (L_1, \dots, L_N) \in (0, +\infty)^N$ with commensurable components such that $L \succcurlyeq \Lambda$, $1 \leq \frac{\Lambda_j}{L_j} < 1 + \varepsilon$ for every $j \in [\![1, N]\!]$, and, for every $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n} \leq T$, one has $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$ if and only if $L \cdot \mathbf{n} = L \cdot \mathbf{n}'$.*

Proof. Write $\Lambda = M\ell$, with $M = \left(m_{jk}\right)_{j \in [\![1,N]\!], k \in [\![1,h]\!]} \in \mathcal{M}_{N,h}(\mathbb{N})$ for some $h \in [\![1, N]\!]$ and $\ell = (\ell_1, \dots, \ell_h) \in (0, +\infty)^h$ with rationally independent components, chosen according to Proposition 4.9. For $n \in \mathbb{N}^*$, we define $L^{(n)} = \left(L_1^{(n)}, \dots, L_N^{(n)}\right) \in [0, +\infty)^N$ by $L^{(n)} = \frac{1}{n} M \lfloor n\ell \rfloor$, where $\lfloor n\ell \rfloor = (\lfloor n\ell_1 \rfloor, \dots, \lfloor n\ell_h \rfloor)$. We claim that $L^{(n)}$ satisfies the required properties for $n \in \mathbb{N}^*$ large enough.

Notice first that, if $n \geq 1/\ell_{\min}$, then all the components of $\lfloor n\ell \rfloor$ are positive, and hence $L^{(n)} \in (0, +\infty)^N$. Moreover, $L^{(n)} \in \mathbb{Q}^N$, and thus $L^{(n)}$ has commensurable components. If $\mathbf{n} \in Z(\Lambda)$, one has $\Lambda \cdot \mathbf{n} = 0$, which yields $\mathbf{n}^{\mathrm{T}} M\ell = 0$ and, since ℓ has rationally independent components and the row vector $\mathbf{n}^{\mathrm{T}} M$ has integer components, one obtains that $\mathbf{n}^{\mathrm{T}} M = 0$, which implies that $L^{(n)} \cdot \mathbf{n} = \frac{1}{n} \mathbf{n}^{\mathrm{T}} M \lfloor n\ell \rfloor = 0$, and hence $\mathbf{n} \in Z(L^{(n)})$, proving that $L^{(n)} \succcurlyeq \Lambda$.

For $j \in [\![1, N]\!]$, since $n\ell_j - 1 < \left\lfloor n\ell_j \right\rfloor \leq n\ell_j$, one obtains from the definition of $L^{(n)}$ that $L_j^{(n)} = \frac{1}{n} \sum_{k=1}^h m_{jk} \lfloor n\ell_k \rfloor \leq \Lambda_j$ and that $L_j^{(n)} \geq \Lambda_j - \frac{1}{n} \sum_{k=1}^h m_{jk} \geq \Lambda_j - \lvert M \rvert_\infty / n$. Hence, for $n \geq 1/\ell_{\min}$, one has $1 \leq \frac{\Lambda_j}{L_j^{(n)}} \leq 1 + \frac{\lvert M \rvert_\infty}{n L_j^{(n)}}$. Notice that, by construction, for every $j \in [\![1, N]\!]$, one has $L_j^{(n)} \to \Lambda_j$ as $n \to +\infty$. Hence there exists $N_1 \geq 1/\ell_{\min}$ such that, for $n \geq N_1$, $L_j^{(n)} \geq \Lambda_j/2$ for every $j \in [\![1, N]\!]$. Thus, for $n \geq N_1$, one has $1 \leq \frac{\Lambda_j}{L_j^{(n)}} \leq 1 + \frac{2|M|_\infty}{n\Lambda_j} \leq 1 + \frac{2|M|_\infty}{n\Lambda_{\min}}$. Letting $N_2 \geq N_1$ be such that $N_2 > \frac{2|M|_\infty}{\varepsilon\Lambda_{\min}}$, one obtains that $1 \leq \frac{\Lambda_j}{L_j^{(n)}} < 1 + \varepsilon$ for every $j \in [\![1, N]\!]$ and $n \geq N_2$.

To prove the last part of the lemma, notice that, for every $n \geq 1/\ell_{\min}$, since $\Lambda \preccurlyeq L^{(n)}$, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$, then $\mathbf{n} - \mathbf{n}' \in Z(\Lambda)$ and thus $L^{(n)} \cdot \mathbf{n} = L^{(n)} \cdot \mathbf{n}'$. Let $\mathcal{F}$ denote the finite set $\mathcal{F} = \{\mathbf{n} \in \mathbb{N}^N \mid \Lambda \cdot \mathbf{n} \leq (1+\varepsilon)T\}$ and define

$$\delta = \min\left\{\left|\Lambda \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}'\right| \,\middle|\, \mathbf{n}, \mathbf{n}' \in \mathcal{F},\ \Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}'\right\} > 0.$$

Since $L^{(n)} \to \Lambda$ as $n \to +\infty$ and $\mathcal{F}$ is finite, there exists $N_3 \geq N_2$ such that, for $n \geq N_3$, one has $\left|L^{(n)} \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}\right| < \frac{\delta}{3}$ for every $\mathbf{n} \in \mathcal{F}$. Let $n \geq N_3$. Assume, to obtain a contradiction, that $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} \leq T$, $\Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}'$, and $L^{(n)} \cdot \mathbf{n} = L^{(n)} \cdot \mathbf{n}'$. Then, using that $1 \leq \frac{\Lambda_j}{L_j^{(n)}} < 1 + \varepsilon$ for every $j \in [\![1, N]\!]$, one computes $\Lambda \cdot \mathbf{n}' < (1+\varepsilon) L^{(n)} \cdot \mathbf{n}' = (1+\varepsilon) L^{(n)} \cdot \mathbf{n} \leq (1+\varepsilon)\Lambda \cdot \mathbf{n} \leq (1+\varepsilon)T$, which shows that $\mathbf{n}' \in \mathcal{F}$. But

$$\delta \leq \left|\Lambda \cdot \mathbf{n} - \Lambda \cdot \mathbf{n}'\right| \leq \left|\Lambda \cdot \mathbf{n} - L^{(n)} \cdot \mathbf{n}\right| + \left|L^{(n)} \cdot \mathbf{n} - L^{(n)} \cdot \mathbf{n}'\right| + \left|L^{(n)} \cdot \mathbf{n}' - \Lambda \cdot \mathbf{n}'\right| < \frac{2\delta}{3},$$

which is a contradiction since $\delta > 0$. Hence, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda \cdot \mathbf{n} \leq T$ and $\Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}'$ one has $L^{(n)} \cdot \mathbf{n} \neq L^{(n)} \cdot \mathbf{n}'$. ∎

Proof of Theorem 5.22. Let $\varepsilon > 0$ and take L as in Lemma 5.23. If $\mathbf{n} \in \mathbb{N}^N$ is such that $\Lambda \cdot \mathbf{n} \leq T$, then $[\mathbf{n}]_\Lambda = [\mathbf{n}]_L$, since it follows from Proposition 5.19 that $[\mathbf{n}]_\Lambda \subset [\mathbf{n}]_L$ and, if $\mathbf{n}' \in [\mathbf{n}]_L$, Lemma 5.23 shows that $\mathbf{n}' \in [\mathbf{n}]_\Lambda$ since $\Lambda \cdot \mathbf{n} \leq T$. In particular, the only equivalence class from $\mathcal{N}_\Lambda$ contained in $[\mathbf{n}]_L$ is $[\mathbf{n}]_\Lambda$. Hence, Proposition 5.19 shows that, for $\mathbf{n} \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n} \leq T$, one has

$$\widehat{\Xi}_{[\mathbf{n}]}^{L} = \sum_{\substack{\tau \in \mathcal{N}_\Lambda \\ \tau \subset [\mathbf{n}]_L}} \widehat{\Xi}_{\tau}^{\Lambda} = \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda},$$

and thus

$$\begin{aligned}\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} &= \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{L} B w \,\middle|\, \mathbf{n} \in \mathbb{N}^N,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} \\ &\subset \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{L} B w \,\middle|\, \mathbf{n} \in \mathbb{N}^N,\ L \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\},\end{aligned}$$

since $L \cdot \mathbf{n} \leq \Lambda \cdot \mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$. Hence relative controllability of $\Sigma(A, B, \Lambda)$ in time T implies relative controllability of $\Sigma(A, B, L)$ in time T. ∎

5.3.3 Minimal time for relative controllability

As stated in Remark 5.14, when $N = 1$ and (5.1) is written as $x(t) = Ax(t - \Lambda) + Bu(t)$, relative controllability in time T is equivalent to Kalman condition $\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda \rfloor}B\end{pmatrix} = d$. Thanks to Cayley–Hamilton Theorem, for every $T \geq (d-1)\Lambda$, one has

$$\operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{\lfloor T/\Lambda \rfloor}B\end{pmatrix} = \operatorname{rk}\begin{pmatrix}B & AB & A^2B & \cdots & A^{d-1}B\end{pmatrix}.$$

Hence, if the system is relatively controllable for some time $T > 0$, it is also relatively controllable in time $T = (d-1)\Lambda$, which proves that its minimal controllability time $T_{\min}$ satisfies $T_{\min} \leq (d-1)\Lambda$. The uniformity of this upper bound on the matrices A and B is important for practical applications, since, if one is interested in finding out whether a given system is relatively controllable for some time $T > 0$, it suffices to verify whether it is relatively controllable in time $T = (d-1)\Lambda$, which can be done algorithmically in a finite number of steps upper bounded by a constant independent of A and B. The goal of this section is to generalize this upper bound on the minimal controllability time $T_{\min}$ for systems with larger N.

We start by considering the case of systems with commensurable delays. In this case, by considering an augmented system in higher dimension, one can characterize the relative controllability of $\Sigma(A, B, \Lambda)$ in terms of a certain output controllability of the augmented system, as shown in the next lemma.

Lemma 5.24. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, and $T > 0$. Assume that Λ has commensurable components and let $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$ be such that $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$. Denote $K = \max_{j \in [\![1,N]\!]} k_j$. Then $\Sigma(A, B, \Lambda)$ is relatively controllable in time $T > 0$ if and only if, for every $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ and $x_1 \in \mathbb{C}^d$, there exists $u : [0, T] \to \mathbb{C}^m$ such that the unique solution $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ of*

$$\begin{cases} X(t) = \widehat{A} X(t-\lambda) + \widehat{B} u(t), & t \in [0, T], \\ X(t) = X_0(t), & t \in [-\lambda, 0), \end{cases} \tag{5.22}$$

satisfies $\widehat{C} X(T) = x_1$, where the matrices $\widehat{A} \in \mathcal{M}_{Kd}(\mathbb{C})$, $\widehat{B} \in \mathcal{M}_{Kd,m}(\mathbb{C})$, and $\widehat{C} \in \mathcal{M}_{d,Kd}(\mathbb{C})$ are given by

$$\begin{aligned} \widehat{A} &= \begin{pmatrix} \widehat{A}_1 & \widehat{A}_2 & \widehat{A}_3 & \cdots & \widehat{A}_K \\ \mathrm{Id}_d & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{Id}_d & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{Id}_d & 0 \end{pmatrix} \in \mathcal{M}_{Kd}(\mathbb{C}), & \widehat{B} &= \begin{pmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{Kd,m}(\mathbb{C}), \\ \widehat{C} &= \begin{pmatrix} \mathrm{Id}_d & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathcal{M}_{d,Kd}(\mathbb{C}), & \widehat{A}_k &= \sum_{\substack{j=1 \\ k_j = k}}^{N} A_j \quad \text{for } k \in [\![1, K]\!], \end{aligned} \tag{5.23}$$

Proof. It is immediate to verify that $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ is the solution of $\Sigma(A, B, \Lambda)$ with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and control $u : [0, T] \to \mathbb{C}^m$ if and only if the function $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ defined by

$$X(t) = \begin{pmatrix} x(t) \\ x(t-\lambda) \\ x(t-2\lambda) \\ \vdots \\ x(t-(K-1)\lambda) \end{pmatrix}$$

is the solution of (5.22) with control u and with initial condition $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ given by

$$X_0(t) = \begin{pmatrix} x_0(t) \\ x_0(t-\lambda) \\ x_0(t-2\lambda) \\ \vdots \\ x_0(t-(K-1)\lambda) \end{pmatrix}.$$

Since $\widehat{C} X(t) = x(t)$ for every $t \in [-\lambda, T]$, the statement of the lemma follows immediately from Theorem 5.12. ∎

Since (5.22) is a controlled difference equation with a single delay, we use Lemma 5.24 to characterize the relative controllability of $\Sigma(A,B,\Lambda)$ in terms of a Kalman rank condition.

Corollary 5.25. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$, and $T>0$. Assume that Λ has commensurable components. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in time T if and only if*

$$\operatorname{rk}\begin{pmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\lfloor T/\lambda\rfloor}\widehat{B}\end{pmatrix}=d, \tag{5.24}$$

where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and λ are as in the statement of Lemma 5.24.

Proof. Notice that, by Proposition 5.8, the solution $X:[-\lambda,T]\to\mathbb{C}^{Kd}$ of (5.22) with initial condition $X_0:[-\lambda,0)\to\mathbb{C}^{Kd}$ and control $u:[0,T]\to\mathbb{C}^m$ is given by

$$X(t)=\widehat{A}^{1+\lfloor t/\lambda\rfloor}X_0\left(t-\left(1+\left\lfloor\frac{t}{\lambda}\right\rfloor\right)\lambda\right)+\sum_{n=0}^{\lfloor t/\lambda\rfloor}\widehat{A}^n\widehat{B}u(t-n\lambda).$$

Hence

$$\widehat{C}X(T)=\widehat{C}\widehat{A}^{1+\lfloor T/\lambda\rfloor}X_0\left(T-\left(1+\left\lfloor\frac{T}{\lambda}\right\rfloor\right)\lambda\right)+\sum_{n=0}^{\lfloor T/\lambda\rfloor}\widehat{C}\widehat{A}^n\widehat{B}u(T-n\lambda). \tag{5.25}$$

If $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then, by Lemma 5.24, taking $X_0=0$, one obtains that, for every $x_1\in\mathbb{C}^d$, there exists $u:[0,T]\to\mathbb{C}^m$ such that $\sum_{n=0}^{\lfloor T/\lambda\rfloor}\widehat{C}\widehat{A}^n\widehat{B}u(T-n\lambda)=x_1$, which shows that (5.24) holds. Conversely, if (5.24) holds, it follows that the matrix $\begin{pmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \cdots & \widehat{C}\widehat{A}^{\lfloor T/\lambda\rfloor}\widehat{B}\end{pmatrix}$ admits a right inverse $M\in\mathcal{M}_{(\lfloor T/\lambda\rfloor+1)m,d}(\mathbb{C})$. For $X_0:[-\lambda,0)\to\mathbb{C}^{Kd}$ and $x_1\in\mathbb{C}^d$, let $U=\left(U_j\right)_{j=0}^{\lfloor T/\lambda\rfloor}\in\mathbb{C}^{(\lfloor T/\lambda\rfloor+1)m}$ be given by

$$U=\begin{pmatrix}U_0\\ \vdots\\ U_{\lfloor T/\lambda\rfloor}\end{pmatrix}=M\left[x_1-\widehat{C}\widehat{A}^{1+\lfloor T/\lambda\rfloor}X_0\left(T-\left(1+\left\lfloor\frac{T}{\lambda}\right\rfloor\right)\lambda\right)\right]$$

and take $u:[0,T]\to\mathbb{C}^m$ satisfying $u(T-n\lambda)=U_n$ for every $n\in[\![0,\lfloor T/\lambda\rfloor]\!]$. It follows immediately from (5.25) that the solution of (5.22) with initial condition X_0 and control u satisfies $\widehat{C}X(T)=x_1$, and hence, by Lemma 5.24, $\Sigma(A,B,\Lambda)$ is relatively controllable in time T. ∎

Thanks to Cayley–Hamiltion Theorem, Corollary 5.25 allows one to obtain an upper bound on the minimal controllability time for $\Sigma(A,B,\Lambda)$ with commensurable delays.

Lemma 5.26. *Let $A=(A_1,\dots,A_N)\in\mathcal{M}_d(\mathbb{C})^N$, $B\in\mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda=(\Lambda_1,\dots,\Lambda_N)\in(0,+\infty)^N$. Assume that Λ has commensurable components. If there exists $T>0$ such that $\Sigma(A,B,\Lambda)$ is relatively controllable in time T, then its minimal controllability time $T_{\min}$ satisfies $T_{\min}\leq(d-1)\Lambda_{\max}$.*

Proof. For $j\in[\![1,K]\!]$, set

$$\widehat{C}_j=\begin{pmatrix}0_{d,(j-1)d} & \mathrm{Id}_d & 0_{d,(K-j)d}\end{pmatrix}\in\mathcal{M}_{d,Kd}(\mathbb{C}).$$

In particular, $\widehat{C}_1=\widehat{C}$. For every $j\in[\![2,K]\!]$, one has $\widehat{C}_j\widehat{A}=\widehat{C}_{j-1}$, and thus $\widehat{C}=\widehat{C}_K\widehat{A}^{K-1}$. Hence, for every $k\in\mathbb{N}$, one has

$$\begin{pmatrix}\widehat{C}\widehat{B} & \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{A}^2\widehat{B} & \cdots & \widehat{C}\widehat{A}^k\widehat{B}\end{pmatrix}=\begin{pmatrix}\widehat{C}_K\widehat{A}^{K-1}\widehat{B} & \widehat{C}_K\widehat{A}^{K}\widehat{B} & \widehat{C}_K\widehat{A}^{K+1}\widehat{B} & \cdots & \widehat{C}_K\widehat{A}^{K+k-1}\widehat{B}\end{pmatrix}.$$

Moreover, since $\widehat{C}_K \widehat{A}^j = \widehat{C}_{K-j}$ for every $j \in [\![0, K-1]\!]$, one computes, for $j \in [\![0, K-2]\!]$, $\widehat{C}_K \widehat{A}^j \widehat{B} = \widehat{C}_{K-j} \widehat{B} = 0$, which shows that

$$\operatorname{rk}\left(\widehat{CB} \quad \widehat{CAB} \quad \widehat{CA^2B} \quad \cdots \quad \widehat{CA^kB}\right) = \operatorname{rk}\left(\widehat{C}_K \widehat{B} \quad \widehat{C}_K \widehat{A}\widehat{B} \quad \widehat{C}_K \widehat{A}^2 \widehat{B} \quad \cdots \quad \widehat{C}_K \widehat{A}^{K+k-1} \widehat{B}\right). \tag{5.26}$$

Let $T > 0$ be such that $\Sigma(A, B, \Lambda)$ is relatively controllable in time T. If $T \leq (d-1)\Lambda_{\max}$, one has immediately that $T_{\min} \leq (d-1)\Lambda_{\max}$. If $T > (d-1)\Lambda_{\max}$, one has, by Corollary 5.25 and (5.26), that

$$\operatorname{rk}\left(\widehat{C}_K \widehat{B} \quad \widehat{C}_K \widehat{A}\widehat{B} \quad \widehat{C}_K \widehat{A}^2 \widehat{B} \quad \cdots \quad \widehat{C}_K \widehat{A}^{K+\lfloor T/\lambda \rfloor - 1} \widehat{B}\right) = d.$$

By Cayley–Hamilton Theorem, since $\widehat{A} \in \mathcal{M}_{Kd}(\mathbb{C})$, this implies that

$$\begin{aligned} d &= \operatorname{rk}\left(\widehat{C}_K \widehat{B} \quad \widehat{C}_K \widehat{A}\widehat{B} \quad \widehat{C}_K \widehat{A}^2 \widehat{B} \quad \cdots \quad \widehat{C}_K \widehat{A}^{K+\lfloor T/\lambda \rfloor - 1} \widehat{B}\right) \\ &= \operatorname{rk}\left(\widehat{C}_K \widehat{B} \quad \widehat{C}_K \widehat{A}\widehat{B} \quad \widehat{C}_K \widehat{A}^2 \widehat{B} \quad \cdots \quad \widehat{C}_K \widehat{A}^{Kd-1} \widehat{B}\right) \end{aligned}$$

since $K + \lfloor T/\lambda \rfloor - 1 \geq Kd - 1$. Hence, by Corollary 5.25 and (5.26), it follows that $\Sigma(A, B, \Lambda)$ is relatively controllable in time $T = K(d-1)\lambda = (d-1)\Lambda_{\max}$, which proves that $T_{\min} \leq (d-1)\Lambda_{\max}$. ■

Now that Lemma 5.26 has established a uniform upper bound on the minimal controllability time for $\Sigma(A, B, \Lambda)$ with commensurate delays, one can use Theorems 5.20 and 5.22 in order to deduce a uniform upper bound for all delay vectors $\Lambda \in (0, +\infty)^N$.

Theorem 5.27. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$. If there exists $T > 0$ such that $\Sigma(A, B, \Lambda)$ is relatively controllable in time T, then its minimal controllability time $T_{\min}$ satisfies $T_{\min} \leq (d-1)\Lambda_{\max}$.*

Proof. Let $\varepsilon > 0$ and choose $L \in (0, +\infty)^N$ according to Theorem 5.22. Then $\Sigma(A, B, L)$ is relatively controllable in time T. Thanks to Lemma 5.26, the minimal controllability time $T_{\min}^{(L)}$ for $\Sigma(A, B, L)$ satisfies $T_{\min}^{(L)} \leq (d-1)L_{\max}$, and, in particular, $\Sigma(A, B, L)$ is relatively controllable in time $(d-1)L_{\max}$. Hence, by Theorem 5.20, $\Sigma(A, B, \Lambda)$ is relatively controllable in time $(1+\varepsilon)(d-1)L_{\max}$, which proves that the minimal controllability time $T_{\min}$ for $\Sigma(A, B, \Lambda)$ satisfies $T_{\min} \leq (1+\varepsilon)(d-1)L_{\max} \leq (1+\varepsilon)(d-1)\Lambda_{\max}$. Since $\varepsilon > 0$ is arbitrary, one concludes that $T_{\min} \leq (d-1)\Lambda_{\max}$. ■

Theorem 5.27 shows that, given $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda \in (0, +\infty)^N$, if one wants to check whether $\Sigma(A, B, \Lambda)$ is relatively controllable in some time $T > 0$, it suffices to verify whether it is relatively controllable in time $(d-1)\Lambda_{\max}$, i.e., if

$$\operatorname{Span}\left\{\Xi_{[\mathbf{n}]}^{\Lambda} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda \cdot \mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d$$

or, equivalently, if

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ \Lambda \cdot \mathbf{n} \leq (d-1)\Lambda_{\max},\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d, \tag{5.27}$$

where $e_1, \dots, e_m$ is the canonical basis of $\mathbb{C}^m$. The set whose span is evaluated in the left-hand side of (5.27) is finite, its cardinality being upper bounded by $m\#\{\mathbf{n} \in \mathbb{N}^N \mid \lvert\mathbf{n}\rvert_1 \leq (d-1)\Lambda_{\max}/\Lambda_{\min}\}$, which is large when $\Lambda_{\max}/\Lambda_{\min}$ is large. The next results provides a way of improving such upper bound, and hence reducing the number of elements to be evaluated in order to study the relative controllability of $\Sigma(A, B, \Lambda)$.

Theorem 5.28. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda, L \in (0, +\infty)^N$ with $\Lambda \preccurlyeq L$. Then $\Sigma(A, B, \Lambda)$ is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L \cdot \mathbf{n} \leq (d-1) L_{\max},\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d. \tag{5.28}$$

Proof. If (5.28) is satisfied, then, since $\Lambda \cdot \mathbf{n} \leq \frac{\Lambda_{\max}}{L_{\min}} L \cdot \mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$, one obtains that

$$\begin{aligned}
\mathbb{C}^d &= \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L \cdot \mathbf{n} \leq (d-1) L_{\max},\ j \in [\![1, m]\!]\right\} \\
&\subset \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} \leq (d-1) \Lambda_{\max} \frac{L_{\max}}{L_{\min}},\ j \in [\![1, m]\!]\right\}
\end{aligned}$$

which proves that $\Sigma(A, B, \Lambda)$ is relatively controllable in time $T = (d-1)\Lambda_{\max} \frac{L_{\max}}{L_{\min}}$ (and also in time $T = (d-1)\Lambda_{\max}$ thanks to Theorem 5.27).

Let $\varepsilon > 0$. Write $\Lambda = M\ell$, with $M \in \mathcal{M}_{N,h}(\mathbb{N})$ for some $h \in [\![1, N]\!]$ and $\ell = (\ell_1, \dots, \ell_h) \in (0, +\infty)^h$ with rationally independent components, chosen according to Proposition 4.9. Since $\Lambda \preccurlyeq L$, it follows from Proposition 4.9 that $L \in \operatorname{Ran} M$, and thus there exists $r \in \mathbb{R}^h$ such that $L = Mr$. Take $r_\varepsilon \in \mathbb{R}^h$ with rationally independent components satisfying $\lvert r - r_\varepsilon\rvert_\infty < \varepsilon / \lvert M\rvert_\infty$, and set $L_\varepsilon = M r_\varepsilon$. Then $\lvert L - L_\varepsilon\rvert_\infty < \varepsilon$ and, in particular, $L_\varepsilon \in (0, +\infty)^N$ for ε small enough. Notice that $L_\varepsilon \approx \Lambda$, since $\Lambda \preccurlyeq L_\varepsilon$ by construction and, if $\mathbf{n} \in \mathbb{N}^N$ is such that $L_\varepsilon \cdot \mathbf{n} = 0$, then $\mathbf{n}^{\mathrm{T}} M r_\varepsilon = 0$, which implies, from the fact that r_ε has rationally independent components and that $\mathbf{n}^{\mathrm{T}} M$ is a row vector of integers, that $\mathbf{n}^{\mathrm{T}} M = 0$, yielding $\Lambda \cdot \mathbf{n} = \mathbf{n}^{\mathrm{T}} M \ell = 0$, and thus $L_\varepsilon \preccurlyeq \Lambda$. Since $\Lambda \approx L_\varepsilon$, it follows from Theorem 5.20 that $\Sigma(A, B, \Lambda)$ is relatively controllable in some time $T > 0$ if and only if $\Sigma(A, B, L_\varepsilon)$ is relatively controllable in some time, i.e.,

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{L_\varepsilon} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{L_\varepsilon},\ L_\varepsilon \cdot \mathbf{n} \leq (d-1) L_{\varepsilon\,\max},\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d.$$

By Proposition 5.19, this is equivalent to

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_{\Lambda},\ L_\varepsilon \cdot \mathbf{n} \leq (d-1) L_{\varepsilon\,\max},\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d. \tag{5.29}$$

Notice that, if ε is small enough, then, for every $\mathbf{n} \in \mathbb{N}^N$, $L_\varepsilon \cdot \mathbf{n} \leq (d-1) L_{\varepsilon\,\max}$ implies $L \cdot \mathbf{n} \leq (d-1) L_{\max}$. Indeed, assume that, for every $\varepsilon > 0$, there exists $\mathbf{n}_\varepsilon \in \mathbb{N}^N$ such that $L_\varepsilon \cdot \mathbf{n}_\varepsilon \leq (d-1) L_{\varepsilon\,\max}$ and $L \cdot \mathbf{n}_\varepsilon > (d-1) L_{\max}$. Then $(d-1) L_{\max} < L \cdot \mathbf{n}_\varepsilon \leq (d-1) L_{\varepsilon\,\max} + (L - L_\varepsilon) \cdot \mathbf{n}_\varepsilon$, which implies that $(d-1) L_{\max} < L \cdot \mathbf{n}_\varepsilon \leq (d-1) L_{\max} + \varepsilon(d - 1 + \lvert \mathbf{n}_\varepsilon\rvert_1)$ and so

$$(d-1) L_{\max} < L \cdot \mathbf{n}_\varepsilon \leq (d-1) L_{\max} + \varepsilon (d-1) \left(1 + \frac{L_{\varepsilon\,\max}}{L_{\varepsilon\,\min}}\right) \tag{5.30}$$

Since the set $\{L \cdot \mathbf{n} \mid \mathbf{n} \in \mathbb{N}^N\} \cap [0, \tau]$ is finite for every $\tau \geq 0$, one obtains that, for every $K \geq 0$, the set $\{\mathbf{n} \in \mathbb{N}^N \mid K < L \cdot \mathbf{n} \leq K + \delta\}$ is empty if $\delta > 0$ is small enough. Hence, since $L_{\varepsilon\,\max}/L_{\varepsilon\,\min} \to L_{\max}/L_{\min}$ as $\varepsilon \to 0$, one obtains that, for $\varepsilon > 0$ small enough, (5.30) cannot be satisfied, which proves that $L_\varepsilon \cdot \mathbf{n} \leq (d-1) L_{\varepsilon\,\max}$ implies $L \cdot \mathbf{n} \leq (d-1) L_{\max}$ for $\varepsilon > 0$ small enough.

If $\Sigma(A, B, \Lambda)$ is relatively controllable in some time, then (5.29) is satisfied. Hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
\mathbb{C}^d &= \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L_\varepsilon \cdot \mathbf{n} \leq (d-1) L_{\varepsilon\,\max},\ j \in [\![1, m]\!]\right\} \\
&\subset \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B e_j \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ L \cdot \mathbf{n} \leq (d-1) L_{\max},\ j \in [\![1, m]\!]\right\},
\end{aligned}$$

which proves (5.28). ∎

Notice that the set whose span is evaluated on the left-hand side of (5.28) has at most $m\#\{\mathbf{n} \in \mathbb{N}^N \mid \lvert\mathbf{n}\rvert_1 \leq (d-1)L_{\max}/L_{\min}\}$ elements, which is an improvement with respect to the upper bound obtained previously for the set whose span is evaluated on the left-hand side of (5.27) as soon as $L_{\max}/L_{\min} < \Lambda_{\max}/\Lambda_{\min}$. Hence Theorem 5.28 allows one to algorithmically check whether $\Sigma(A,B,\Lambda)$ is relatively controllable in less steps than by using (5.27). In particular, since we have $\Lambda \preccurlyeq (1,1,\dots,1)$ for every $\Lambda \in (0,+\infty)^N$ with rationally independent components, one obtains the following improvement of (5.27) in this case.

Corollary 5.29. *Let $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda \in (0,+\infty)^N$. Assume that Λ has rationally independent components. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{\Xi_{\mathbf{n}} B e_j \,\middle|\, \mathbf{n} \in \mathbb{N}^N,\ \lvert\mathbf{n}\rvert_1 \leq d-1,\ j \in [\![1,m]\!]\right\} = \mathbb{C}^d.$$

In the case $\Lambda \preccurlyeq (1,1,\dots,1)$, one can also provide an alternative proof for Theorem 5.27 and Corollary 5.29, which, instead of relying on the augmented system from Lemma 5.24 and the approximation argument in the proof of Theorem 5.28, uses rather a technique quite similar to the proof of Theorem 4.36. Due to the interesting features of such alternative proof, we provide it in Appendix 5.A.

Remark 5.30. The statements and proofs of the results from this section and the previous one can be slightly modified to show that, for every $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1,\dots,\Lambda_N) \in (0,+\infty)^N$, and $T \geq (d-1)\Lambda_{\max}$, one has

$$\begin{aligned} &\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} \\ &= \operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m\right\}. \end{aligned}$$

The set $\mathsf{V} = \operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m\right\}$ is the set of all states $x_1 \in \mathbb{C}^d$ that can be reached by the system $\Sigma(A,B,\Lambda)$ after time $T \geq (d-1)\Lambda_{\max}$ starting from a zero initial condition.

When $N = 1$ and the controlled difference equation (5.1) becomes $x(t) = Ax(t-\Lambda) + Bu(t)$ with $A = A_1$ and $\Lambda = \Lambda_1$, Kalman decomposition (see, e.g., [163, Lemma 3.3.3]) states that there exists $P \in \mathrm{GL}_d(\mathbb{C})$ such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

with $A_{11} \in \mathcal{M}_r(\mathbb{C})$, $A_{22} \in \mathcal{M}_{d-r}(\mathbb{C})$, $B_1 \in \mathcal{M}_{r,m}(\mathbb{C})$, where $r = \dim \mathsf{V}$, the pair (A_{11}, B_1) is controllable, and $P\mathsf{V} = \mathbb{C}^r \times \{0\}^{d-r} = \operatorname{Span}\{e_1,\dots,e_r\}$.

Such decomposition does not hold for larger N in general, i.e., one cannot find in general, for $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda \in (0,+\infty)^N$ for which $\Sigma(A,B,\Lambda)$ is not relatively controllable in any time $T > 0$, a matrix $P \in \mathrm{GL}_d(\mathbb{C})$ for which one would have, for every $j \in [\![1,N]\!]$,

$$PA_jP^{-1} = \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ 0 & A_{22}^{(j)} \end{pmatrix}, \qquad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tag{5.31}$$

with $A_{11}^{(j)} \in \mathcal{M}_r(\mathbb{C})$, $A_{22}^{(j)} \in \mathcal{M}_{d-r}(\mathbb{C})$, $B_1 \in \mathcal{M}_{r,m}(\mathbb{C})$, with $r \in [\![1,d-1]\!]$ and such that $\Sigma(A_{11}^{(1)},\dots,A_{11}^{(N)}, B_1, \Lambda)$ is relatively controllable in time $T \geq (r-1)\Lambda_{\max}$. Indeed, consider the case $N = 2$,

$d = 4$, $m = 1$, $\Lambda = (1, \ell)$ for some $\ell \in \left(\frac{3}{4}, 1\right)$, and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & \pi & 0 & 1 \\ -3 & \sqrt{2} & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 \\ 0 & \pi+1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sqrt{3} & 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice that

$$\begin{aligned} &\operatorname{Span}\{\Xi_{\mathbf{n}} B \mid \mathbf{n} = (n_1, n_2) \in \mathbb{N}^2,\ n_1 + \ell n_2 \leq 3\} \\ &= \operatorname{Span}\{\Xi_{(0,0)}B, \Xi_{(0,1)}B, \Xi_{(0,2)}B, \Xi_{(0,3)}B, \Xi_{(1,0)}B, \Xi_{(1,1)}B, \Xi_{(1,2)}B, \Xi_{(2,0)}B, \Xi_{(2,1)}B, \Xi_{(3,0)}B\} \\ &= \operatorname{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi+3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi^2+4\pi+7 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi+3 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ \pi^2+4\pi+7 \\ (5+\pi)\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}\right\} = \{0\} \times \mathbb{C}^3, \end{aligned}$$

and thus, by the definition of relative controllability and Theorem 5.27, one obtains that $\Sigma(A, B, \Lambda)$ is not relatively controllable in any time $T > 0$. We claim that this system cannot be decomposed under the form (5.31). If it were the case, one immediately verifies from (5.31) that the vector space $\mathsf{V} = P^{-1}(\mathbb{C}^r \times \{0\}^{4-r})$ would contain B and be invariant under left multiplication by A_1 and A_2. Such invariance implies in particular that $\Xi_{\mathbf{n}} B \in \mathsf{V}$ for every $\mathbf{n} \in \mathbb{N}^2$, and thus $\{0\} \times \mathbb{C}^3 \subset \mathsf{V}$. Such invariance then also implies that

$$\mathsf{V} \ni A_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \pi \\ \sqrt{2} \end{pmatrix},$$

which shows that $\mathsf{V} = \mathbb{C}^4$, contradicting the fact that $\mathsf{V} = P^{-1}(\mathbb{C}^r \times \{0\}^{4-r})$ for $P \in \mathrm{GL}_4(\mathbb{C})$ and $r \in [\![1, 3]\!]$. Hence $\Sigma(A, B, \Lambda)$ cannot be put under the form (5.31).

5.4 Exact and approximate controllability in L^2

This section considers the problem of the exact and approximate controllability of the state $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0)}$ of (5.1) in the function space $L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$. We start with the notations that will be used here.

Definition 5.31. Let $T \in (0, +\infty)$. We define the Hilbert spaces X and Y_T by $\mathsf{X} = L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$ and $\mathsf{Y}_T = L^2((0, T), \mathbb{C}^m)$ endowed with their usual inner products.

Recall that, thanks to Remark 5.3, if $x_0 \in \mathsf{X}$ and $u \in \mathsf{Y}_T$, then the unique solution x of (5.1) satisfies $x_t \in \mathsf{X}$ for every $t \in [0, T]$. We now provide the definitions of the controllability notions used in this section.

Definition 5.32. Let $T \in (0, +\infty)$.

(a) We say that (5.1) is *exactly controllable in time* T if, for every $x_0, \overline{x} \in \mathsf{X}$, there exists $u \in \mathsf{Y}_T$ such that the solution x of (5.1) with initial condition x_0 and control u satisfies $x_T = \overline{x}$.

(b) We say that (5.1) is *approximately controllable in time* T if, for every $x_0, \overline{x} \in \mathsf{X}$ and $\varepsilon > 0$, there exists $u \in \mathsf{Y}_T$ such that the solution x of (5.1) with initial condition x_0 and control u satisfies $\|x_T - \overline{x}\|_{\mathsf{X}} < \varepsilon$.

(c) We define the bounded linear operator $E(T) : \mathsf{Y}_T \to \mathsf{X}$ by

$$(E(T)u)(t) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{n}\leq T+t}} \Xi_{\mathbf{n}} Bu(T + t - \Lambda\cdot\mathbf{n}). \tag{5.32}$$

Exact or approximate controllability in time T implies the same kind of controllability for every time $T' \geq T$, since one can take a control u equal to zero in the interval $(0, T'-T)$ and control the system from $T'-T$ until T'. Moreover, one clearly has that exact controllability in time T implies approximate controllability in time T.

A useful result for studying exact and approximate controllability is the following lemma, which states that such properties are preserved under linear change of coordinates, linear feedback, and changes of the time scale.

Lemma 5.33. *Let $T > 0$, $\lambda > 0$, $K_j \in \mathcal{M}_{m,d}(\mathbb{C})$ for $j \in [\![1, N]\!]$, $P \in \mathrm{GL}_d(\mathbb{C})$, and consider the system*

$$x(t) = \sum_{j=1}^{N} P(A_j + BK_j)P^{-1} x\left(t - \frac{\Lambda_j}{\lambda}\right) + PBu(t). \tag{5.33}$$

Then

(a) (5.1) *is exactly controllable in time T if and only if* (5.33) *is exactly controllable in time $\frac{T}{\lambda}$;*

(b) (5.1) *is approximately controllable in time T if and only if* (5.33) *is approximately controllable in time $\frac{T}{\lambda}$.*

Proof. Let us prove (b), the proof of (a) being similar. Assume that (5.1) is approximately controllable in time T and take $x_0, \overline{x} \in L^2((-\Lambda_{\max}/\lambda, 0), \mathbb{C}^d)$ and $\varepsilon > 0$. Let $\widetilde{x}_0, \widetilde{\overline{x}} \in L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$ be given by $\widetilde{x}_0(t) = P^{-1}x_0(t/\lambda)$ and $\widetilde{\overline{x}}(t) = P^{-1}\overline{x}(t/\lambda)$. Since (5.1) is approximately controllable in time T, there exists $\widetilde{u} \in L^2((0,T), \mathbb{C}^m)$ such that the solution $\widetilde{x}$ of (5.1) with initial condition $\widetilde{x}_0$ and control $\widetilde{u}$ satisfies $\left\lVert \widetilde{x}_T - \widetilde{\overline{x}} \right\rVert_{L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)} < \frac{\varepsilon\sqrt{\lambda}}{\lvert P\rvert_2}$. Let $u \in L^2((0, T/\lambda), \mathbb{C}^m)$ and $x \in L^2((-\Lambda_{\max}/\lambda, T/\lambda), \mathbb{C}^d)$ be given by

$$u(t) = \widetilde{u}(\lambda t) - \sum_{j=1}^{N} K_j \widetilde{x}(\lambda t - \Lambda_j), \qquad x(t) = P\widetilde{x}(\lambda t).$$

A straightforward computation shows that x is the solution of (5.33) with initial condition x_0 and control u, and that $x_{T/\lambda}(t) = P\widetilde{x}_T(\lambda t)$ for $t \in (-\Lambda_{\max}/\lambda, 0)$. Hence one has that $\left\lVert x_{T/\lambda} - \overline{x} \right\rVert_{L^2((-\Lambda_{\max}/\lambda, 0), \mathbb{C}^d)} < \varepsilon$, and thus (5.33) is approximately controllable in time $\frac{T}{\lambda}$. The converse is proved in a similar way. ∎

The operator $E(T)$ maps a control u to the corresponding solution at time T of (5.1) with initial condition 0. It follows immediately from Proposition 5.8 that, for every $T > 0$, $x_0 \in \mathsf{X}$, and $u \in \mathsf{Y}_T$, the corresponding solution x of (5.1) satisfies

$$x_T = S(T)x_0 + E(T)u, \tag{5.34}$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup defined in Remark 5.10. Equation (5.34) allows one to immediately obtain the following classical characterization of exact and approximate controllability in terms of the operator $E(T)$ (cf. [55, Lemma 2.46]).

Proposition 5.34. *Let $T \in (0, +\infty)$.*

(a) *System* (5.1) *is exactly controllable in time* T *if and only if* $E(T)$ *is surjective.*

(b) *System* (5.1) *is approximately controllable in time* T *if and only if* $\operatorname{Ran} E(T)$ *is dense in* X.

We recall in the next proposition the classical characterizations of exact and approximate controllability in terms of the adjoint operator $E(T)^*$, whose proofs can be found, e.g., in [55, Section 2.3.2].

Proposition 5.35. *Let* $T \in (0, +\infty)$.

(a) *System* (5.1) *is exactly controllable in time* T *if and only if there exists* $c > 0$ *such that, for every* $x \in \mathsf{X}$,
$$\|E(T)^* x\|_{\mathsf{Y}_T}^2 \geq c\|x\|_{\mathsf{X}}^2.$$

(b) *System* (5.1) *is approximately controllable in time* T *if and only if* $E(T)^*$ *is injective, i.e., for every* $x \in \mathsf{X}$,
$$E(T)^* x = 0 \implies x = 0.$$

In order to apply Proposition 5.35, we provide in the next lemma an explicit formula for $E(T)^*$, which can be obtained directly from the definition of adjoint operator.

Lemma 5.36. *Let* $T \in (0, +\infty)$. *The adjoint operator* $E(T)^* : \mathsf{X} \to \mathsf{Y}_T$ *is given by*
$$(E(T)^* x)(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ -\Lambda_{\max} \leq t - T + \Lambda \cdot \mathbf{n} < 0}} B^* \Xi_{\mathbf{n}}^* x(t - T + \Lambda \cdot \mathbf{n}). \tag{5.35}$$

Proof. Let $u \in \mathsf{Y}_T$, $x \in \mathsf{X}$. In order to simplify the notations, we extend x and u by zero outside their intervals of definition. We have
$$\begin{aligned}
\langle E(T)u, x\rangle_{\mathsf{X}} &= \int_{-\Lambda_{\max}}^{0} \sum_{\mathbf{n} \in \mathbb{N}^N} \langle \Xi_{\mathbf{n}} B u(T + s - \Lambda \cdot \mathbf{n}), x(s)\rangle_{\mathbb{C}^d} \, ds \\
&= \sum_{\mathbf{n} \in \mathbb{N}^N} \int_{-\Lambda_{\max}}^{0} u(T + s - \Lambda \cdot \mathbf{n})^* B^* \Xi_{\mathbf{n}}^* x(s) ds \\
&= \sum_{\mathbf{n} \in \mathbb{N}^N} \int_{T - \Lambda \cdot \mathbf{n} - \Lambda_{\max}}^{T - \Lambda \cdot \mathbf{n}} u(t)^* B^* \Xi_{\mathbf{n}}^* x(t - T + \Lambda \cdot \mathbf{n}) dt \\
&= \int_0^T \left\langle u(t), \sum_{\mathbf{n} \in \mathbb{N}^N} B^* \Xi_{\mathbf{n}}^* x(t - T + \Lambda \cdot \mathbf{n})\right\rangle_{\mathbb{C}^m} dt,
\end{aligned}$$
where we use that the above infinite sums have only finitely many non-zero terms. ■

Remark 5.37. One can provide a graphical representation for the operators $E(T)$ and $E(T)^*$ as follows. In a plane with coordinates (ξ, ζ), we draw in the domain $[0, T) \times [-\Lambda_{\max}, 0)$, for $\mathbf{n} \in \mathbb{N}^N$, the line segment $\sigma_{\mathbf{n}}$ defined by the equation $\zeta = \xi - T + \Lambda \cdot \mathbf{n}$ (see Figure 5.1). We associate with the line segment $\sigma_{\mathbf{n}}$ the matrix coefficient $\Xi_{\mathbf{n}} B$.

For $u \in \mathsf{Y}_T$, (5.32) can be interpreted as follows. For $s \in [-\Lambda_{\max}, 0)$, we draw the horizontal line $\zeta = s$. Each intersection between this line and a line segment $\sigma_{\mathbf{n}}$ gives one term in the sum for $(E(T)u)(s)$. This term consists of the matrix coefficient corresponding to the line $\sigma_{\mathbf{n}}$ multiplied by u evaluated at the ξ-coordinate of the intersection point.

Similarly, for $x \in \mathsf{X}$, (5.35) can be interpreted as follows. For $t \in [0, T)$, we draw the vertical line $\xi = t$. As before, each intersection between this line and a line segment $\sigma_{\mathbf{n}}$ gives one term in the sum for $(E(T)^* x)(t)$. This term consists of the Hermitian transpose of the matrix coefficient corresponding to the line $\sigma_{\mathbf{n}}$ multiplied by x evaluated at the ζ-coordinate of the intersection point.

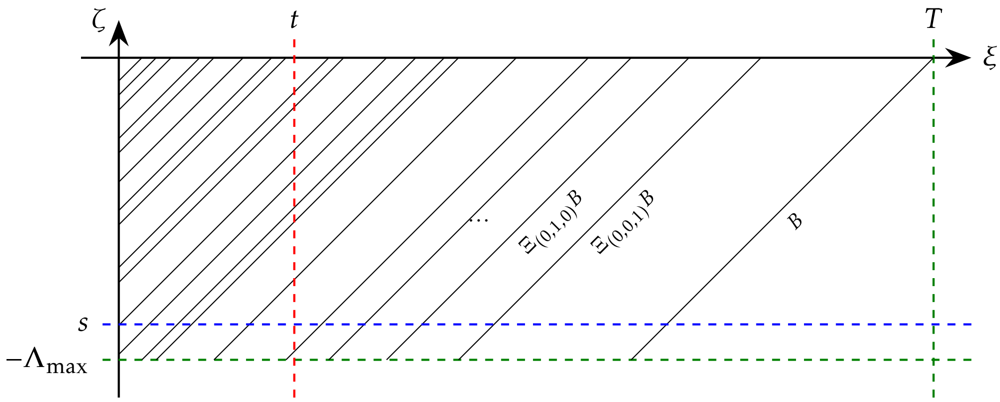


Figure 5.1: Graphical representation for $E(T)$ and $E(T)^*$ in the case $N = 3$, $\Lambda_1 = 2$, $\Lambda_2 = \frac{\sqrt{5}+1}{2}$, $\Lambda_3 = \pi - 2$, and $T = e^2 - 2$. The matrix coefficients associated with the line segments $\sigma_{\mathbf{n}}$ are given in the picture for $\mathbf{n} = (0,0,0)$, $\mathbf{n} = (0,0,1)$, and $\mathbf{n} = (0,1,0)$.

5.4.1 Commensurable delays

We consider in this section the problem of characterizing exact and approximate controllability of (5.1) in the case where the delays $\Lambda_1, \dots, \Lambda_N$ are commensurable. A classical procedure in such situation is to perform an augmentation of the state of the system — as it was done in Lemma 5.24 to study relative controllability — to obtain an equivalent system with a single delay, whose controllability can be easily characterized using Kalman criterion for discrete-time systems. For the sake of completeness, we detail such approach in Lemma 5.38 and Proposition 5.40. An important limitation of this technique is that it cannot be generalized to the case where $\Lambda_1, \dots, \Lambda_N$ are not assumed to be commensurable.

Proposition 5.34 provides another approach to the controllability of (5.1), through the range of the operator $E(T)$. In this section, we also characterize the operator $E(T)$ in Lemma 5.46 in order to obtain a controllability criterion for (5.1) in Proposition 5.47. It turns out that this criterion is the same as the one from Proposition 5.40, as we prove in Theorem 5.49.

The main goal of this section is thus to show that studying the controllability of (5.1) for commensurable delays $\Lambda_1, \dots, \Lambda_N$ through the operator $E(T)$ leads to the same controllability criterion as the classical approach of augmenting the state of the system. However, differently from the latter, the operator $E(T)$ can be defined regardless of the commensurability of $\Lambda_1, \dots, \Lambda_N$.

Let us first consider the augmentation of the state of (5.1). The next lemma, whose proof is straightforward, provides the construction of the augmented state and the difference equation it satisfies.

Lemma 5.38. *Let $T \in (0, +\infty)$, $u : [0, T] \to \mathbb{C}^m$, and suppose that $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$ with $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$. Let $K = \max_{j \in [\![1,N]\!]} k_j$.*

(a) *If $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ is the solution of (5.1) with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, then the function $X : [-\lambda, T) \to \mathbb{C}^{Kd}$ defined by*

$$X(t) = \begin{pmatrix} x(t) \\ x(t-\lambda) \\ x(t-2\lambda) \\ \vdots \\ x(t-(K-1)\lambda) \end{pmatrix} \tag{5.36}$$

satisfies

$$X(t) = \widehat{A} X(t - \lambda) + \widehat{B} u(t) \tag{5.37}$$

with $\widehat{A}$ and $\widehat{B}$ given by (5.23) and with initial condition $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ given by

$$X_0(t) = \begin{pmatrix} x_0(t) \\ x_0(t-\lambda) \\ x_0(t-2\lambda) \\ \vdots \\ x_0(t-(K-1)\lambda) \end{pmatrix}. \tag{5.38}$$

(b) *If $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ is the solution of (5.37) with initial condition $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$, with $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ given by (5.23), then the function $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ defined by*

$$x(t) = \begin{cases} \widehat{C} X(t), & \text{if } t \in [0, T], \\ x_0(t), & \text{if } t \in [-\Lambda_{\max}, 0), \end{cases}$$

is the solution of (5.1) with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, where x_0 is the unique function satisfying (5.38) for every $t \in [-\lambda, 0)$.

Remark 5.39. Lemma 5.38 considers solutions of (5.1) and (5.37) in the sense of Definition 5.1, i.e., with no regularity assumptions. However, one immediately obtains from (5.36) that, for every $t \in [0, T]$, $x_t \in \mathsf{X}$ if and only if $X_t \in L^2((-\lambda, 0), \mathbb{C}^{Kd})$, and in this case $\|x_t\|_{\mathsf{X}} = \|X_t\|_{L^2((-\lambda,0),\mathbb{C}^{Kd})}$.

As an immediate consequence of Lemma 5.38, we obtain the following criterion.

Proposition 5.40. *Let $T \in (0, +\infty)$ and suppose that $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$ with $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$. Let $K = \max_{j \in [\![1,N]\!]} k_j$ and define $\widehat{A}$ and $\widehat{B}$ from $A_1, \dots, A_N, B$ as in (5.23). Then the following assertions are equivalent.*

(a) *System (5.1) is exactly controllable in time T;*

(b) *System (5.1) is approximately controllable in time T;*

(c) *$T \geq (\kappa + 1)\lambda$, where $\kappa = \inf\left\{n \in \mathbb{N} \;\middle|\; \operatorname{rk}\begin{pmatrix} \widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B} & \cdots & \widehat{A}^n\widehat{B} \end{pmatrix} = Kd\right\} \in \mathbb{N} \cup \{\infty\}$.*

Proof. Notice first that the solution $X : [-\lambda, T] \to \mathbb{C}^{Kd}$ of (5.37) with initial condition $X_0 : [-\lambda, 0) \to \mathbb{C}^{Kd}$ and control $u : [0, T] \to \mathbb{C}^m$ is given by

$$X(t) = \widehat{A}^{1 + \lfloor t/\lambda \rfloor} X_0\left(t - \left(1 + \left\lfloor \frac{t}{\lambda} \right\rfloor\right)\lambda\right) + \sum_{n=0}^{\lfloor t/\lambda \rfloor} \widehat{A}^n \widehat{B} u(t - n\lambda). \tag{5.39}$$

We will prove that (a) $\implies$ (b) $\implies$ (c) $\implies$ (a). The first implication is trivial due to the definitions of exact and approximate controllability. Suppose now that (b) holds, let $M = \left\lfloor \frac{T}{\lambda} \right\rfloor$, $\rho = (M+1)\lambda - T > 0$, take $w \in \mathbb{C}^{Kd}$ and $\varepsilon > 0$, and write $w = \left(w_1^{\mathrm{T}}, \dots, w_K^{\mathrm{T}}\right)^{\mathrm{T}}$ with $w_1, \dots, w_K \in \mathbb{C}^d$. Let $\overline{x} \in \mathsf{X}$ be defined by the relations $\overline{x}(t) = w_j$ for $t \in [-j\lambda, -(j-1)\lambda)$, $j \in [\![1, K]\!]$. By (b), there exists $u \in \mathsf{Y}_T$ such that the solution x of (5.1) with zero initial condition and control u satisfies $\|x_T - \overline{x}\|_{\mathsf{X}} < \rho\varepsilon$. Defining $X \in L^2((-\lambda, T), \mathbb{C}^{Kd})$ by (5.36), we obtain that $\|X_T - w\|_{L^2((-\lambda,0),\mathbb{C}^{Kd})} < \rho\varepsilon$. Using Lemma 5.38 and (5.39), we obtain that

$$\int_{T-\lambda}^{M\lambda} \left|\sum_{n=0}^{M-1} \widehat{A}^n \widehat{B} u(t - n\lambda) - w\right|_{\mathbb{C}^{Kd}}^2 dt \leq \int_{T-\lambda}^{T} \left|\sum_{n=0}^{\lfloor t/\lambda \rfloor} \widehat{A}^n \widehat{B} u(t - n\lambda) - w\right|_{\mathbb{C}^{Kd}}^2 dt < \rho\varepsilon,$$

and, in particular, there exists a set of positive measure $J \subset (T-\lambda, M\lambda)$ such that

$$\left|\sum_{n=0}^{M-1} \widehat{A}^n \widehat{B} u(t-n\lambda) - w\right|_{\mathbb{C}^{Kd}}^2 < \varepsilon$$

for $t \in J$. Hence, we have shown that, for every $w \in \mathbb{C}^{Kd}$ and $\varepsilon > 0$, there exist $u_0, \dots, u_{M-1} \in \mathbb{C}^m$ such that $\left|\sum_{n=0}^{M-1} \widehat{A}^n \widehat{B} u_n - w\right|^2 < \varepsilon$, which in particular implies that $M \geq 1$. This proves that the range of the matrix $\begin{pmatrix}\widehat{B} & \widehat{AB} & \widehat{A}^2 B & \cdots & \widehat{A}^{M-1}\widehat{B}\end{pmatrix} \in \mathcal{M}_{Kd,Mm}(\mathbb{C})$ is dense in $\mathbb{C}^{Kd}$, and hence is equal to $\mathbb{C}^{Kd}$, leading to $\kappa \leq M-1$ by definition of κ. Thus $T \geq M\lambda \geq (\kappa+1)\lambda$, which proves (c).

Assume now that (c) holds. In particular, since $T < +\infty$, one has $\kappa \in \mathbb{N}$. We will prove the exact controllability of (5.1) in time $T_0 = (\kappa+1)\lambda$, which in particular implies its exact controllability in time T. Let $x_0, \overline{x} \in \mathsf{X}$. Define $X_0, \overline{X} \in L^2((-\lambda, 0), \mathbb{C}^{Kd})$ from $x_0, \overline{x}$ respectively as in (5.38). Let $C = \begin{pmatrix}\widehat{B} & \widehat{AB} & \cdots & \widehat{A}^\kappa \widehat{B}\end{pmatrix} \in \mathcal{M}_{Kd,(\kappa+1)m}(\mathbb{C})$, which, by (c), has full rank, and thus admits a right inverse $C^\# \in \mathcal{M}_{(\kappa+1)m,Kd}(\mathbb{C})$. Let $u \in \mathsf{Y}_{T_0}$ be the unique function defined by the relation

$$\begin{pmatrix} u(t+(\kappa+1)\lambda) \\ u(t+\kappa\lambda) \\ \vdots \\ u(t+\lambda) \end{pmatrix} = C^\# \left(\overline{X}(t) - \widehat{A}^{\kappa+1} X_0(t)\right) \qquad \text{for almost every } t \in (-\lambda, 0).$$

A straightforward computation shows, together with (5.39), that the unique solution X of (5.37) with initial condition X_0 and control u satisfies $X_{T_0} = \overline{X}$, and hence, by Lemma 5.38, the unique solution of (5.1) with initial condition x_0 and control u satisfies $x_{T_0} = \overline{x}$, which proves (a). ∎

Remark 5.41. A first important consequence of Proposition 5.40 is that exact and approximate controllability are equivalent for systems with commensurable delays. As it follows from the results in Section 5.4.2, this is no longer true without the commensurability hypothesis.

Remark 5.42. It follows from Cayley–Hamilton theorem that κ from Proposition 5.40 is either infinite or belongs to $[\![0, Kd-1]\!]$. In particular, (c) is satisfied for some $T \in (0, +\infty)$ if and only if the controllability matrix $\mathcal{C}(\widehat{A}, \widehat{B}) \in \mathcal{M}_{Kd,Kdm}(\mathbb{C})$ has full rank. Moreover, condition (c) is satisfied for some $T \in (0, +\infty)$ if and only if it is satisfied for every $T \in [(\kappa+1)\lambda, +\infty)$, and thus (exact or approximate) controllability in time $T \geq (\kappa+1)\lambda$ is equivalent to (the same kind of) controllability in time $T = (\kappa+1)\lambda$.

Remark 5.43. When $m = 1$, it follows from the definition of κ that $\kappa \geq Kd - 1$ and thus, from Remark 5.42, $\kappa \in \{Kd-1, +\infty\}$. It follows that a system with a single input is either (exactly and approximately) controllable in time $T = d\Lambda_{\max}$ or not controllable in any time $T \in (0, +\infty)$.

In the remainder of this section, we characterize the controllability of (5.1) using the operator $E(T)$ from (5.32) instead of the augmented system from Lemma 5.38.

Definition 5.44. Let $T \in (0, +\infty)$ and suppose that $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$ with $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$. Let $K = \max_{j \in [\![1,N]\!]} k_j$, $M = \left\lfloor \frac{T}{\lambda} \right\rfloor$, and $\delta = T - \lambda M \in [0, \lambda)$. We define the operators $R_1 : \mathsf{X} \to L^2((-\lambda,0),\mathbb{C}^d)^K$ and $R_2 : \mathsf{Y}_T \to L^2((-\lambda,0),\mathbb{C}^m)^M \times L^2((-\delta,0),\mathbb{C}^m)$ by

$$(R_1 x(t))_n = x(t-(n-1)\lambda), \qquad \text{for } t \in (-\lambda,0) \text{ and } n \in [\![1,K]\!],$$

$$(R_2 u(t))_n = u(t+T-(n-1)\lambda), \qquad \text{for } \begin{cases} t \in (-\lambda,0) \text{ if } n \in [\![1,M]\!], \\ t \in (-\delta,0) \text{ if } n = M+1. \end{cases}$$

Remark 5.45. It follows immediately from the definitions of R_1 and R_2 that these operators are unitary transformations.

Lemma 5.46. *Let $T \in (0,+\infty)$ and suppose that $(\Lambda_1,\dots,\Lambda_N) = \lambda(k_1,\dots,k_N)$ with $\lambda > 0$ and $k_1,\dots,k_N \in \mathbb{N}^*$. Let K, M, δ, R_1, and R_2 be as in Definition 5.44. Then, for every $u \in L^2((-\lambda,0),\mathbb{C}^m)^M \times L^2((-\delta,0),\mathbb{C}^m)$,*

$$R_1 E(T) R_2^{-1} u = C P_1 u + \mathrm{E} P_2 u,$$

where $P_1 : L^2((-\lambda,0),\mathbb{C}^m)^M \times L^2((-\delta,0),\mathbb{C}^m) \to L^2((-\lambda,0),\mathbb{C}^m)^M$ is the projection in the first M coordinates, $P_2 : L^2((-\lambda,0),\mathbb{C}^m)^M \times L^2((-\delta,0),\mathbb{C}^m) \to L^2((-\lambda,0),\mathbb{C}^m)$ is the projection in the last coordinate composed with an extension by zero in the interval $(-\lambda,-\delta)$, and $C \in \mathcal{M}_{Kd,Mm}(\mathbb{C}), \mathrm{E} \in \mathcal{M}_{Kd,m}(\mathbb{C})$ are given by

$$\begin{aligned} C &= \left(C_{j\ell}\right)_{j\in[\![1,K]\!],\ell\in[\![1,M]\!]}, & C_{j\ell} &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell-j}} \Xi_{\mathbf{n}} B \quad \text{for } j \in [\![1,K]\!],\ \ell \in [\![1,M]\!], \\ \mathrm{E} &= \left(\mathrm{E}_j\right)_{j\in[\![1,K]\!]}, & \mathrm{E}_j &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=M+1-j}} \Xi_{\mathbf{n}} B \quad \text{for } j \in [\![1,K]\!]. \end{aligned} \tag{5.40}$$

Proof. Let $u \in \mathsf{Y}_T$ and extend u by zero in the interval $(-\infty,0)$. From (5.32) and Definition 5.44, we have that, for $j \in [\![1,K]\!]$ and $t \in (-\lambda,0)$,

$$\begin{aligned} (R_1 E(T) u(t))_j &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ \Lambda\cdot\mathbf{n}\le T+t-(j-1)\lambda}} \Xi_{\mathbf{n}} B u(t+T-\Lambda\cdot\mathbf{n}-(j-1)\lambda) \\ &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}\le \frac{T+t}{\lambda}-(j-1)}} \Xi_{\mathbf{n}} B u(t+T-(k\cdot\mathbf{n}+j-1)\lambda) \\ &= \sum_{\ell=1}^{M} \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell-j}} \Xi_{\mathbf{n}} B u(t+T-(\ell-1)\lambda) + \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=M+1-j}} \Xi_{\mathbf{n}} B u(t+T-M\lambda) \\ &= \sum_{\ell=1}^{M} C_{j\ell} (P_1 R_2 u(t))_\ell + \mathrm{E}_j (P_2 R_2 u(t)), \end{aligned}$$

which gives the required result. ∎

Proposition 5.47. *Let $T \in (0,+\infty)$ and suppose that $(\Lambda_1,\dots,\Lambda_N) = \lambda(k_1,\dots,k_N)$ with $\lambda > 0$ and $k_1,\dots,k_N \in \mathbb{N}^*$. Let K, M, and $C \in \mathcal{M}_{Kd,Mm}(\mathbb{C})$ be as in Lemma 5.46. Then the following assertions are equivalent.*

(a) *System (5.1) is exactly controllable in time T;*

(b) *System (5.1) is approximately controllable in time T;*

(c) *The matrix C has full rank.*

Proof. We will prove that (a) $\implies$ (b) $\implies$ (c) $\implies$ (a). The first implication is trivial. Suppose now that (b) holds, which means, from Proposition 5.34(b), that $\operatorname{Ran} E(T)$ is dense in X. Since R_1 and R_2 are unitary transformations, Lemma 5.46 shows that the range of the operator $CP_1 + \mathrm{E}P_2 : L^2((-\lambda, 0), \mathbb{C}^m)^M \times L^2((-\delta, 0), \mathbb{C}^m) \to L^2((-\lambda, 0), \mathbb{C}^d)^K$ is also dense. Let $\Pi : L^2((-\lambda, 0), \mathbb{C}^d)^K \to L^2((-\lambda, -\delta), \mathbb{C}^d)^K$ be the restriction to the non-empty interval $(-\lambda, -\delta)$, which is surjective. Thus the range of $\Pi(CP_1 + \mathrm{E}P_2)$ is dense in $L^2((-\lambda, -\delta), \mathbb{C}^d)^K$, and one has, from the definition of Π and P_2, that $\Pi\mathrm{E}P_2 = 0$, which shows that the range of ΠCP_1 is dense in $L^2((-\lambda, -\delta), \mathbb{C}^d)^K$. But $(\Pi CP_1 u(t))_j = \sum_{\ell=1}^M C_{j\ell} u_\ell(t)$ for every $u \in L^2((-\lambda, 0), \mathbb{C}^m)^M \times L^2((-\delta, 0), \mathbb{C}^m)$, $j \in [\![1, M]\!]$, and $t \in (-\lambda, -\delta)$, and hence the density of the range of ΠCP_1 in $L^2((-\lambda, -\delta), \mathbb{C}^d)^K$ implies that C has full rank, which proves (c).

Suppose now that (c) holds. Then the operator $CP_1 : L^2((-\lambda, 0), \mathbb{C}^m)^M \times L^2((-\delta, 0), \mathbb{C}^m) \to L^2((-\lambda, 0), \mathbb{C}^d)^K$ is surjective, which implies, using Lemma 5.46 and the fact that R_1 and R_2 are unitary transformations, that $E(T)$ is surjective. Thus, by Proposition 5.34(a), (5.1) is exactly controllable in time T. ∎

Remark 5.48. One can use the graphical representation of $E(T)$ from Remark 5.37 to construct the matrices C and E from Lemma 5.46. Indeed, when $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$ for some $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$, one can consider a grid in $[0, T) \times [-\Lambda_{\max}, 0)$ defined by the horizontal lines $\zeta = -j\lambda$, $j \in [\![1, K]\!]$, and by the vertical lines $\xi = T - (\ell - 1)\lambda$, $\ell \in [\![1, M+1]\!]$, where $K = \max_{j \in [\![1, N]\!]} k_j$ and $M = \left\lfloor \frac{T}{\lambda} \right\rfloor$. This grid contains square cells $S_{j\ell} = (T - \ell\lambda, T - (\ell - 1)\lambda) \times (-j\lambda, -(j-1)\lambda)$ for $j \in [\![1, K]\!]$, $\ell \in [\![1, M+1]\!]$, and rectangular cells $R_j = (0, T - M\lambda) \times (-j\lambda, -(j-1)\lambda)$, the latter being empty when T is an integer multiple of λ (see Figure 5.2).

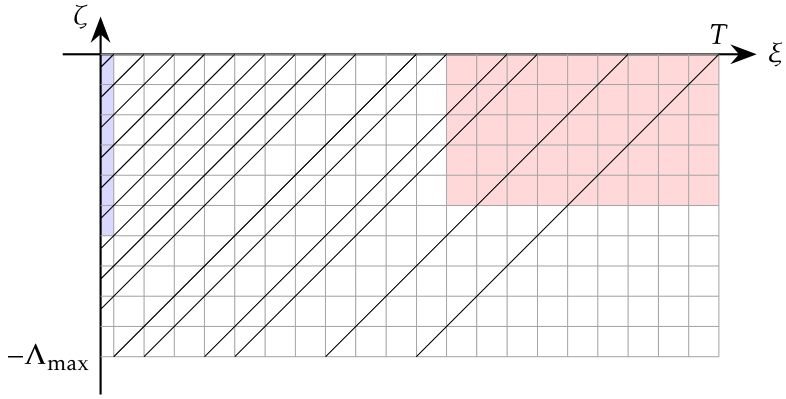


Figure 5.2: Graphical representation for $E(T)$ in the case $N = 3$, $\Lambda = \left(1, \frac{7}{10}, \frac{3}{10}\right)$, $\lambda = \frac{1}{10}$, and $T \in (2, 2 + \lambda)$.

Consider the line segments $\sigma_{\mathbf{n}}$ from Remark 5.37. Due to the commensurability of the delays $\Lambda_1, \dots, \Lambda_N$, the intersection between each line segment $\sigma_{\mathbf{n}}$ and a square $S_{j\ell}$ is either empty or equal to the diagonal of the square from its bottom-left to its top-right edge, and, similarly, the intersection between each $\sigma_{\mathbf{n}}$ and a rectangle R_j is either empty or equal to a line segment starting at the top-right edge of the rectangle. The matrix $C = \left(C_{j\ell}\right)_{j \in [\![1, K]\!],\, \ell \in [\![1, M]\!]}$ can thus be constructed as follows. For $j \in [\![1, K]\!]$ and $\ell \in [\![1, M]\!]$, the matrix $C_{j\ell}$ is the sum over all $\mathbf{n} \in \mathbb{N}^N$ such that $\sigma_{\mathbf{n}}$ intersects the square $S_{j\ell}$ of the matrix coefficients corresponding to such $\sigma_{\mathbf{n}}$. Similarly, $\mathrm{E} = \left(\mathrm{E}_j\right)_{j \in [\![1, K]\!]}$ is constructed by defining, for $j \in [\![1, K]\!]$, E_j as the sum over all $\mathbf{n} \in \mathbb{N}^N$ such that $\sigma_{\mathbf{n}}$ intersects the rectangle R_j of the matrix coefficients corresponding to such $\sigma_{\mathbf{n}}$.

Figure 5.2 represents this construction in the case $N = 3$, $\Lambda = \left(1, \frac{7}{10}, \frac{3}{10}\right)$, $\lambda = \frac{1}{10}$, and $T \in (2, 2+\lambda)$. The first $5d$ lines and $9m$ columns of the matrix C are

$$C = \begin{pmatrix} B & 0 & 0 & \Xi_{(0,0,1)}B & 0 & 0 & \Xi_{(0,0,2)}B & \Xi_{(0,1,0)}B & 0 & \cdots \\ 0 & B & 0 & 0 & \Xi_{(0,0,1)}B & 0 & 0 & \Xi_{(0,0,2)}B & \Xi_{(0,1,0)}B & \ddots \\ 0 & 0 & B & 0 & 0 & \Xi_{(0,0,1)}B & 0 & 0 & \Xi_{(0,0,2)}B & \ddots \\ 0 & 0 & 0 & B & 0 & 0 & \Xi_{(0,0,1)}B & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & B & 0 & 0 & \Xi_{(0,0,1)}B & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

and the first $6d$ lines of E are

$$\mathrm{E} = \begin{pmatrix} \left(\Xi_{(2,0,0)} + \Xi_{(1,1,1)} + \Xi_{(0,2,2)}\right)B \\ \left(\Xi_{(1,0,3)} + \Xi_{(0,1,4)}\right)B \\ \Xi_{(0,0,6)}B \\ \left(\Xi_{(1,1,0)} + \Xi_{(0,2,1)}\right)B \\ \left(\Xi_{(1,0,2)} + \Xi_{(0,1,3)}\right)B \\ \Xi_{(0,0,5)}B \\ \vdots \end{pmatrix}$$

where the square cells leading to the first $5d$ lines and $9m$ columns of C are highlighted in Figure 5.2, as well as the rectangular cells leading to the first $6d$ lines of E. Notice, in particular, that C is a block-Toeplitz matrix, which is clear from its definition in (5.40).

Propositions 5.40 and 5.47 provide two criteria for the controllability of (5.1) for commensurable delays $\Lambda_1, \dots, \Lambda_N$. The first one is obtained by the usual augmentation of the state and corresponds to a Kalman condition on the augmented matrices $\widehat{A}$ and $\widehat{B}$ from (5.23), whereas the second one uses the characterizations of controllability in terms of the operator $E(T)$ from Proposition 5.34 in order to provide a criterion in terms of the matrix C constructed from the matrix coefficients $\Xi_{\mathbf{n}}B$. The main result of this section is that both criteria are actually the same.

Theorem 5.49. *Let $T \in (0, +\infty)$ and assume that $(\Lambda_1, \dots, \Lambda_N) = \lambda(k_1, \dots, k_N)$ with $\lambda > 0$ and $k_1, \dots, k_N \in \mathbb{N}^*$. Let K, $\widehat{A}$, $\widehat{B}$ be as in Proposition 5.40 and M, C as in Proposition 5.47. Then*

$$C = \begin{pmatrix} \widehat{B} & \widehat{A}\widehat{B} & \widehat{A}^2\widehat{B} & \cdots & \widehat{A}^{M-1}\widehat{B} \end{pmatrix}.$$

Proof. For $j \in [\![1, K]\!]$ and $\ell \in [\![1, M]\!]$, let $C_{j\ell}$ be defined as in (5.40) and set $C_\ell = \left(C_{j\ell}\right)_{j \in [\![1, K]\!]} \in \mathcal{M}_{Kd,m}(\mathbb{C})$. We will prove the theorem by showing that $C_1 = \widehat{B}$ and that $C_{\ell+1} = \widehat{A} C_\ell$ for $\ell \in [\![1, M-1]\!]$.

By (5.40), $C_{j1} = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ k \cdot \mathbf{n} = 1-j}} \Xi_{\mathbf{n}} B$ for $j \in [\![1, K]\!]$, and thus, since $\Xi_{\mathbf{n}} = 0$ for $\mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N$, we obtain that $C_{j1} = 0$ for $j \in [\![2, K]\!]$ and $C_{11} = \Xi_0 B = B$, which shows that $C_1 = \widehat{B}$.

Let $\ell \in [\![1, M-1]\!]$. For $j \in [\![2, K]\!]$, we have $C_{j,\ell+1} = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell+1-j}} \Xi_{\mathbf{n}} B = C_{j-1,\ell} = \left(\widehat{A} C_\ell\right)_j$. Moreover, it follows from (5.5) that

$$
\begin{aligned}
C_{1,\ell+1} &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell}} \Xi_{\mathbf{n}} B = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell}} \sum_{j=1}^{N} A_j \Xi_{\mathbf{n}-e_j} B = \sum_{m=1}^{K} \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell}} \sum_{\substack{j=1 \\ k_j=m}}^{N} A_j \Xi_{\mathbf{n}-e_j} B \\
&= \sum_{m=1}^{K} \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ k\cdot\mathbf{n}=\ell-m}} \sum_{\substack{j=1 \\ k_j=m}}^{N} A_j \Xi_{\mathbf{n}} B = \sum_{m=1}^{K} \widehat{A}_m C_{m\ell} = \left(\widehat{A} C_\ell\right)_1,
\end{aligned}
$$

where $\widehat{A}_m$ is defined as in (5.23). Hence $\widehat{A} C_\ell = C_{\ell+1}$, as required. ∎

Remark 5.50. Lemma 5.46 shows that, when $\Lambda_1, \dots, \Lambda_N$ are commensurable, the operator $E(T)$ can be represented by the matrices E and C, and Proposition 5.47 shows that the controllability of (5.1) is encoded only in the matrix C. The representation of $E(T)$ by the matrix C is also highlighted in Remark 5.48. Hence, the fact that C coincides with the Kalman matrix $\begin{pmatrix}\widehat{B} & \widehat{A}\widehat{B} & \cdots & \widehat{A}^{M-1}\widehat{B}\end{pmatrix}$ for the augmented system (5.37) shows that $E(T)$ generalizes the Kalman matrix for difference equations without the commensurability hypothesis on the delays.

5.4.2 Two-dimensional systems with two delays

Section 5.4.1 presented a controllability criterion for difference equations under the assumption of commensurability of the delays. It is also interesting to investigate the controllability of (5.1) without such assumption. However, this is a much more subtle problem, since the technique of state augmentation from Lemma 5.38 in order to obtain an equivalent system with a single delay cannot be applied without the commensurability hypothesis, and a deeper analysis of the operator $E(T)$ is necessary to study the controllability of (5.1). In this section, we carry out such analysis in the particular case $N = d = 2$ and $m = 1$, obtaining necessary and sufficient conditions for exact and approximate controllability. This simple-looking low-dimensional case already presents several non-trivial features that illustrate the difficulties stemming from the non-commensurability of the delays, including the fact that, contrarily to Propositions 5.40 and 5.47, exact and approximate controllability are no longer equivalent.

Consider the difference equation

$$x(t) = A_1 x(t - \Lambda_1) + A_2 x(t - \Lambda_2) + B u(t), \tag{5.41}$$

where $x(t) \in \mathbb{C}^2$, $u(t) \in \mathbb{C}$, $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, and $B \in \mathcal{M}_{2,1}(\mathbb{C})$, the latter set being canonically identified with $\mathbb{C}^2$. Without loss of generality, we assume that $\Lambda_1 > \Lambda_2$. The main result of this section is the following controllability criterion.

Theorem 5.51. *Let $T \in (0, +\infty)$ and $(\Lambda_1, \Lambda_2) \in (0, +\infty)^2$ with $\Lambda_1 > \Lambda_2$.*

(a) *If (A_1, B) is not controllable, then (5.41) is neither exactly nor approximately controllable in time T.*

(b) *If (A_1, B) is controllable and (A_2, B) is not controllable, then the following are equivalent.*

 (i) *System (5.41) is exactly controllable in time T.*

(ii) *System* (5.41) *is approximately controllable in time* T.

(iii) $T \geq 2\Lambda_1$.

(c) *If* (A_1, B) *and* (A_2, B) *are controllable, take* $Z \in \mathbb{C}^2 \setminus \operatorname{Span}\{B\}$ *and set*

$$\beta = \frac{\det \mathcal{C}(A_1, B)}{\det \mathcal{C}(A_2, B)}, \qquad \alpha = \frac{\det\begin{pmatrix} B & (A_1 - \beta A_2) Z\end{pmatrix}}{\det\begin{pmatrix} B & Z\end{pmatrix}}. \tag{5.42}$$

Then α *does not depend on* Z. *Let* $C \subset \mathbb{C}$ *be the set of all possible complex values of the expression* $\beta + \alpha^{1 - \frac{\Lambda_2}{\Lambda_1}}$.

(i) *System* (5.41) *is exactly controllable in time* T *if and only if* $T \geq 2\Lambda_1$ *and* $0 \notin \overline{C}$.

(ii) *System* (5.41) *is approximately controllable in time* T *if and only if* $T \geq 2\Lambda_1$ *and* $0 \notin C$.

The remainder of this section is dedicated to the proof of Theorem 5.51. The first step is the following characterization of the numbers α, β defined in (5.42).

Lemma 5.52. *Let* $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, $Z \in \mathbb{C}^2 \setminus \operatorname{Span}\{B\}$, *assume that* (A_1, B) *and* (A_2, B) *are controllable, and let* α, β *be given by* (5.42). *Let*

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.43}$$

Then $(A_1 - \beta A_2, B)$ *is not controllable,* B *is a right eigenvector of* $A_1 - \beta A_2$, *and* α *is an eigenvalue of* $A_1 - \beta A_2$ *associated with the left eigenvector* $B^{\mathrm{T}} R$. *In particular,* α *does not depend on* Z.

Proof. Notice that $\det \mathcal{C}(A_1 - \beta A_2, B) = \det\begin{pmatrix} B & (A_1 - \beta A_2) B\end{pmatrix} = \det\begin{pmatrix} B & A_1 B\end{pmatrix} - \beta \det\begin{pmatrix} B & A_2 B\end{pmatrix} = 0$ by definition of β, and thus $(A_1 - \beta A_2, B)$ is not controllable. Moreover, since one has $\det\begin{pmatrix} B & (A_1 - \beta A_2) B\end{pmatrix} = 0$, the vectors $(A_1 - \beta A_2) B$ and B are colinear, and thus $(A_1 - \beta A_2) B = \lambda B$ for some $\lambda \in \mathbb{C}$. Finally, notice that, for every $X, Y \in \mathcal{M}_{2,1}(\mathbb{C})$, $\det\begin{pmatrix} X & Y\end{pmatrix} = X^{\mathrm{T}} R Y$, and thus

$$B^{\mathrm{T}} R (A_1 - \beta A_2) Z = \alpha B^{\mathrm{T}} R Z. \tag{5.44}$$

Moreover, one has $B^{\mathrm{T}} R B = \det\begin{pmatrix} B & B\end{pmatrix} = 0$ and $B^{\mathrm{T}} R (A_1 - \beta A_2) B = \lambda B^{\mathrm{T}} R B = 0$, which shows in particular that $B^{\mathrm{T}} R (A_1 - \beta A_2) B = \alpha B^{\mathrm{T}} R B$. Together with (5.44), this gives $B^{\mathrm{T}} R (A_1 - \beta A_2)(a Z + b B) = \alpha B^{\mathrm{T}} R (a Z + b B)$ for every $a, b \in \mathbb{C}$, which shows that

$$B^{\mathrm{T}} R (A_1 - \beta A_2) = \alpha B^{\mathrm{T}} R$$

and thus $B^{\mathrm{T}} R$ is a left eigenvector of $A_1 - \beta A_2$ associated with the eigenvalue α. ■

We next show, thanks to the characterization of α, β from Lemma 5.52, that α and β are invariant under linear change of variables and linear feedbacks. Before proving this fact in the following lemma, recall that, for any pair of matrices $(A, B) \in \mathcal{M}_d(\mathbb{C}) \times \mathcal{M}_{d,m}(\mathbb{C})$, the controllability of (A, B) implies the controllability of $(P(A + BK)P^{-1}, PB)$ for every $P \in \mathrm{GL}_d(\mathbb{C})$ and $K \in \mathcal{M}_{m,d}(\mathbb{C})$. This classical result from the theory of linear control systems follows from the fact that $\mathcal{C}(P(A + BK)P^{-1}, PB) = P\mathcal{C}(A, B)$, which can be verified by a straightforward computation.

Lemma 5.53. *Let $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, $Z \in \mathbb{C}^2 \setminus \operatorname{Span}\{B\}$, $P \in \mathrm{GL}_2(\mathbb{C})$, $K_1, K_2 \in \mathcal{M}_{1,2}(\mathbb{C})$, and set*

$$\widetilde{B} = PB, \qquad \widetilde{A}_j = P(A_j + BK_j)P^{-1} \quad \text{for } j \in \{1,2\}.$$

Suppose that (A_1, B) and (A_2, B) are controllable. Let $\alpha, \beta \in \mathbb{C}$ be defined by (5.42) and define $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{C}$ by

$$\widetilde{\beta} = \frac{\det \mathcal{C}(\widetilde{A}_1, \widetilde{B})}{\det \mathcal{C}(\widetilde{A}_2, \widetilde{B})}, \qquad \widetilde{\alpha} = \frac{\det\begin{pmatrix}\widetilde{B} & (\widetilde{A}_1 - \widetilde{\beta}\widetilde{A}_2)\widetilde{Z}\end{pmatrix}}{\det\begin{pmatrix}\widetilde{B} & \widetilde{Z}\end{pmatrix}}$$

for some $\widetilde{Z} \in \mathbb{C}^2 \setminus \operatorname{Span}\{B\}$. Then $\widetilde{\alpha} = \alpha$ and $\widetilde{\beta} = \beta$.

Proof. Since $\mathcal{C}(\widetilde{A}_j, \widetilde{B}) = P\mathcal{C}(A_j, B)$ for $j \in \{1,2\}$, one obtains immediately from the definitions of β and $\widetilde{\beta}$ that $\widetilde{\beta} = \beta$. Let R be given by (5.43). By Lemma 5.52, α is an eigenvalue of $A_1 - \beta A_2$ associated with the left eigenvector $B^{\mathrm{T}} R$ and $\widetilde{\alpha}$ is an eigenvalue of $\widetilde{A}_1 - \widetilde{\beta}\widetilde{A}_2$ associated with the left eigenvector $\widetilde{B}^{\mathrm{T}} R$. Using that $(PB)^{\mathrm{T}} R (PB) = 0$ and that $P^{\mathrm{T}} R P = (\det P) R$, we get

$$\begin{aligned}\widetilde{B}^{\mathrm{T}} R(\widetilde{A}_1 - \widetilde{\beta}\widetilde{A}_2) &= B^{\mathrm{T}} P^{\mathrm{T}} R P\left((A_1 - \beta A_2) + B(K_1 - \beta K_2)\right) P^{-1} = (\det P) B^{\mathrm{T}} R (A_1 - \beta A_2) P^{-1} \\ &= \alpha (\det P) B^{\mathrm{T}} R P^{-1} = \alpha B^{\mathrm{T}} P^{\mathrm{T}} R P P^{-1} = \alpha \widetilde{B}^{\mathrm{T}} R,\end{aligned}$$

which shows that $\widetilde{\alpha} = \alpha$. ∎

Remark 5.54. Let $\mathcal{F}, \mathcal{G}$ be two sets with $\mathcal{F} \subset \mathcal{G} \subset \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in (\mathcal{M}_2(\mathbb{C}))^2 \times \mathcal{M}_{2,1}(\mathbb{C}) \times (0, +\infty)^2 \mid \Lambda_1 > \Lambda_2\}$ and satisfying the following condition: for every $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}$, there exists $P \in \mathrm{GL}_2(\mathbb{C})$, $K_1, K_2 \in \mathcal{M}_{1,2}(\mathbb{C})$, and $\lambda > 0$ such that $(P(A_1 + BK_1)P^{-1}, P(A_2 + BK_2)P^{-1}, PB, \lambda\Lambda_1, \lambda\Lambda_2) \in \mathcal{F}$. It follows from Lemmas 5.33 and 5.53 that it suffices to prove Theorem 5.51 for every $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{F}$ in order to obtain its conclusions for every $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}$.

In particular, in order to prove Theorem 5.51 for every $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, and $\Lambda_1, \Lambda_2 \in (0, +\infty)$ with $\Lambda_1 > \Lambda_2$, it suffices to prove it for

$$A_j = \begin{pmatrix} a_{j1} & a_{j2} \\ 0 & 0 \end{pmatrix} \text{ for } j \in \{1,2\}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (\Lambda_1, \Lambda_2) = (1, L) \tag{5.45}$$

with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1,2\}$ and $L \in (0,1)$. Indeed, given $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, and $\Lambda_1, \Lambda_2 \in (0, +\infty)$ with $\Lambda_1 > \Lambda_2$, it suffices to take $\lambda = 1/\Lambda_1$, $P \in \mathrm{GL}_2(\mathbb{C})$ satisfying $PB = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$, and, for $j \in \{1,2\}$, $K_j \in \mathcal{M}_{1,2}(\mathbb{C})$ such that $-K_j P^{-1}$ is equal to the second row of PA_jP^{-1}, and in this case $P(A_1 + BK_1)P^{-1}$, $P(A_2 + BK_2)P^{-1}$, PB, and $(\lambda\Lambda_1, \lambda\Lambda_2)$ are under the form (5.45).

We will thus prove Theorem 5.51 for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}$, with

$$\mathcal{G} = \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in (\mathcal{M}_2(\mathbb{C}))^2 \times \mathcal{M}_{2,1}(\mathbb{C}) \times (0, +\infty)^2 \mid A_1, A_2, B, \text{ and } (\Lambda_1, \Lambda_2) \text{ satisfy (5.45)}\}.$$

Our strategy is to decompose $\mathcal{G}$ into four sets, according to the three parts (a), (b), and (c) of Theorem 5.51. We set

$$\begin{aligned}
\mathcal{G}_{a1} &= \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G} \mid (A_1, B) \text{ and } (A_2, B) \text{ are not controllable}\}, \\
\mathcal{G}_{a2} &= \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G} \mid (A_1, B) \text{ is not controllable and } (A_2, B) \text{ is controllable}\}, \\
\mathcal{G}_{b} &= \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G} \mid (A_1, B) \text{ is controllable and } (A_2, B) \text{ is not controllable}\}, \\
\mathcal{G}_{c} &= \{(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G} \mid (A_1, B) \text{ and } (A_2, B) \text{ are controllable}\}.
\end{aligned}$$

For $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_{a1}$, since $a_{j2} = -\det \mathcal{C}(A_j, B)$ for $j \in \{1, 2\}$, it follows that $a_{12} = a_{22} = 0$, and thus every $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_{a1}$ is of the form

$$A_j = \begin{pmatrix} a_{j1} & 0 \\ 0 & 0 \end{pmatrix} \text{ for } j \in \{1, 2\}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (\Lambda_1, \Lambda_2) = (1, L). \tag{5.46}$$

Concerning $\mathcal{G}_{a2}$, let $\mathcal{F}_{a2} \subset \mathcal{G}_{a2}$ be the set of A_1, A_2, B, and (Λ_1, Λ_2) under the form

$$A_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (\Lambda_1, \Lambda_2) = (1, L). \tag{5.47}$$

Then it suffices to prove Theorem 5.51 for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{F}_{a2}$ in order to obtain its results for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_{a2}$. Indeed, if $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_{a2}$, then, since (A_2, B) is controllable, there exists $P \in GL_2(\mathbb{C})$ and $K_2 \in \mathcal{M}_{1,2}(\mathbb{C})$ such that $P(A_2 + BK_2)P^{-1}$, PB are under the form (5.47) (see, e.g., [163, Definition 5.1.5]). It now suffices to take $K_1 \in \mathcal{M}_{1,2}(\mathbb{C})$ such that $-K_1 P^{-1}$ is equal to the second row of PA_1P^{-1} and one immediately obtains that $P(A_1 + BK_1)P^{-1}$, $P(A_2 + BK_2)P^{-1}$, PB, and (Λ_1, Λ_2) are under the form (5.47). Notice that the coefficient in the first row and second column of A_1 is equal to zero since it must be equal to $-\det \mathcal{C}(A_1, B)$.

Similarly, for $\mathcal{G}_b$, we consider the set $\mathcal{F}_b \subset \mathcal{G}_b$ of all A_1, A_2, B, and (Λ_1, Λ_2) under the form

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_{21} & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (\Lambda_1, \Lambda_2) = (1, L). \tag{5.48}$$

As before, it suffices to prove Theorem 5.51 for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{F}_b$ in order to obtain its results for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_b$.

Finally, for $\mathcal{G}_c$, we consider the set $\mathcal{F}_c \subset \mathcal{G}_c$ of all A_1, A_2, B, and (Λ_1, Λ_2) under the form

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (\Lambda_1, \Lambda_2) = (1, L). \tag{5.49}$$

It also suffices to prove Theorem 5.51 for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{F}_c$ in order to obtain its results for $(A_1, A_2, B, \Lambda_1, \Lambda_2) \in \mathcal{G}_c$. Moreover, by a straightforward computation, one obtains in this case $\alpha = a_{11}$, $\beta = a_{12}$.

We now prove parts (a) and (b) of Theorem 5.51.

Proof of Theorem 5.51(a) and (b). Suppose that (A_1, B) and (A_2, B) are not controllable. According to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.46). Hence one immediately computes

$$\Xi_{\mathbf{n}} B = \begin{cases} B & \text{if } \mathbf{n} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$ and $t \in (-1, 0)$, one has $(E(T)u)(t) = Bu(T + t)$ if $T + t \geq 0$ and $(E(T)u)(t) = 0$ if $T + t < 0$. In particular, the range of $E(T)$ is contained in the set $L^2((-1, 0), \operatorname{Ran} B)$, which is not dense in X. Hence the system is neither exactly nor approximately controllable in any time $T > 0$.

Assume now that (A_1, B) is not controllable and (A_2, B) is controllable, in which case, according to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.47). Hence

$$\Xi_{\mathbf{n}} B = \begin{cases} B & \text{if } \mathbf{n} = 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \mathbf{n} = (0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $u \in \mathsf{Y}_T$, one has

$$(E(T)u)(t) = \begin{cases} 0 & \text{if } -1 \leq T+t < 0, \\ \begin{pmatrix} 0 \\ u(T+t) \end{pmatrix} & \text{if } 0 \leq T+t < L, \\ \begin{pmatrix} u(T+t-L) \\ u(T+t) \end{pmatrix} & \text{if } T+t \geq L. \end{cases} \tag{5.50}$$

If $T < 1+L$, then, for every $u \in \mathsf{Y}_T$, the first component of $E(T)u$ vanishes in the non-empty interval $(-1, L-T)$, and hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T < 1+L$. If $T \geq 1+L$, then, for every $u \in \mathsf{Y}_T$, if $x = E(T)u = (x_1, x_2)$, we have $x_1(t) = u(T+t-L)$ and $x_2(t) = u(T+t)$ for every $t \in (-1,0)$, which implies that $x_2(t) = x_1(t+L)$ for $t \in (-1,-L)$. Hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T \geq 1+L$ either. This concludes the proof of (a).

Concerning (b), assume that (A_1, B) is controllable and (A_2, B) is not controllable. According to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.48). One computes

$$\Xi_{\mathbf{n}} B = \begin{cases} B & \text{if } \mathbf{n} = 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \mathbf{n} = (1,0), \\ 0 & \text{otherwise.} \end{cases} \tag{5.51}$$

Hence, for every $u \in \mathsf{Y}_T$,

$$(E(T)u)(t) = \begin{cases} 0 & \text{if } -1 \leq T+t < 0, \\ \begin{pmatrix} 0 \\ u(T+t) \end{pmatrix} & \text{if } 0 \leq T+t < 1, \\ \begin{pmatrix} u(T+t-1) \\ u(T+t) \end{pmatrix} & \text{if } T+t \geq 1. \end{cases} \tag{5.52}$$

If $T < 2$, then, for every $u \in \mathsf{Y}_T$, the first component of $E(T)u$ vanishes in the non-empty interval $(-1, 1-T)$, and hence the range of $E(T)$ is not dense in X, which shows that the system is neither exactly nor approximately controllable in time $T < 2$. If $T \geq 2$, the system is exactly controllable. Indeed, take $x \in \mathsf{X}$ and write $x = (x_1, x_2)$. Define $u \in \mathsf{Y}_T$ by

$$u(t) = \begin{cases} x_2(t-T), & \text{if } T-1 \leq t < T, \\ x_1(t-T+1), & \text{if } T-2 \leq t < T-1, \\ 0, & \text{otherwise.} \end{cases}$$

Such u is clearly well-defined and, by (5.52), one immediately has that $E(T)u = x$. Hence $E(T)$ is surjective, and thus the system is exactly controllable. ■

Let us now turn to the proof of Theorem 5.51(c). Notice first that, for A_1, A_2, B, and (Λ_1, Λ_2) under the form (5.49), a straightforward computation shows that

$$\Xi_{(n,m)} B = \begin{cases} B & \text{if } n = m = 0, \\ \begin{pmatrix} \alpha^{n-1}\beta \\ 0 \end{pmatrix} & \text{if } m = 0,\, n \geq 1, \\ \begin{pmatrix} \alpha^{n} \\ 0 \end{pmatrix} & \text{if } m = 1, \\ 0 & \text{if } m \geq 2, \end{cases} \tag{5.53}$$

where one uses that $\alpha = a_{11}$ and $\beta = a_{12}$. Hence, for every $u \in \mathsf{Y}_T$, $(E(T)u)(t) = 0$ for $T + t \in (-1, 0)$ and, for $T + t \geq 0$,

$$(E(T)u)(t) = \begin{pmatrix} \displaystyle\sum_{n=0}^{\lfloor T+t-1 \rfloor} \alpha^n \beta u(T + t - n - 1) + \sum_{n=0}^{\lfloor T+t-L \rfloor} \alpha^n u(T + t - n - L) \\ u(T + t) \end{pmatrix}. \tag{5.54}$$

Moreover, for every $x = (x_1, x_2) \in \mathsf{X}$ and $t \in (-T, 0)$, one computes from (5.35) that

$$(E(T)^* x)(t + T) = \begin{cases} x_2(t), & \text{if } -L < t < 0, \\ x_2(t) + x_1(t + L), & \text{if } -1 < t < -L, \\ \overline{\alpha}^{-\lfloor t \rfloor - 2} \overline{\beta} x_1(\{t\} - 1) + \overline{\alpha}^{-\lfloor t+L \rfloor - 1} x_1(\{t + L\} - 1), & \text{if } t < -1. \end{cases} \tag{5.55}$$

With (5.54) and (5.55), one can prove Theorem 5.51(c) in the case $T < 2\Lambda_1$.

Proof of Theorem 5.51(c) for $T < 2\Lambda_1$. Assume that (A_1, B) and (A_2, B) are controllable, in which case, according to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.49), and thus $E(T)$ and $E(T)^*$ are given by (5.54) and (5.55), respectively.

If $T < 1 + L$, it follows from (5.54) that, for every $u \in \mathsf{Y}_T$, the first component of $E(T)u$ vanishes in the non-empty interval $(-1, L - T)$, and hence the system is neither exactly nor approximately controllable in time $T < 1 + L$.

For $1 + L \leq T < 2$, we will show that approximate controllability does not hold (and hence that exact controllability does not hold either) by showing that $E(T)^*$ is not injective. For $x = (x_1, x_2) \in \mathsf{X}$, it follows from (5.55) that $E(T)^* x = 0$ in Y_T if and only if

$$\begin{cases} x_2(t) = 0, & -L < t < 0, \\ x_2(t) + x_1(t + L) = 0, & -1 < t < -L, \\ \overline{\beta} x_1(t + 1 - L) + x_1(t) = 0, & -1 < t < -1 + L, \\ \overline{\beta} x_1(t - L) + \overline{\alpha} x_1(t) = 0, & 1 + L - T < t < 0. \end{cases} \tag{5.56}$$

Since the first two equations of (5.56) define x_2 uniquely in terms of x_1, showing that $E(T)^* x = 0$ for some nonzero function $x \in \mathsf{X}$ amounts to showing that there exists $y \in L^2((-1, 0), \mathbb{C})$ nonzero such that

$$\begin{cases} \overline{\beta} y(t + 1 - L) + y(t) = 0, & -1 < t < -1 + L, \\ \overline{\beta} y(t - L) + \overline{\alpha} y(t) = 0, & 1 + L - T < t < 0. \end{cases} \tag{5.57}$$

Define $f : [-1, 0) \to [-1, 0)$ by $f(t) = t + 1 - L$ if $-1 \leq t < L - 1$ and $f(t) = t - L$ if $L - 1 \leq t < 0$; notice that f is a translation of $1 - L$ modulo 1. For $n \in \mathbb{N}$, set $t_n = f^n(-1)$ and let $K = \min\{n \in \mathbb{N} \mid f^{n+1}(-1) \in [-1, 1 - T)\}$. K is clearly well-defined: if L is rational, all orbits of f are periodic and hence $K + 1$ is upper bounded by the period of the orbit starting at -1, and, if L is irrational, all orbits of f are dense in $[-1, 0)$ and hence they intersect $[-1, 1 - T)$ infinitely many times. Moreover, all the points $t_0, \dots, t_K$ are distinct. For $n \in [\![0, K]\!]$, we define $\gamma_n \in \mathbb{C}$ inductively as follows. We set $\gamma_0 = 1$ and, for $n \in [\![1, K]\!]$, we set $\gamma_n = -\frac{\gamma_{n-1}}{\beta}$ if $-1 \leq t_{n-1} < L - 1$ and $\gamma_n = -\frac{\alpha \gamma_{n-1}}{\beta}$ if $L - 1 \leq t_{n-1} < 0$.

Take $\delta > 0$ small enough such that all the intervals $(t_n, t_n + \delta)$, $n \in [\![0, K]\!]$, are pairwise disjoint, contained in $(-1, 0)$, and do not contain any of the points $1 - T$, $L - 1$, $1 + L - T$, and $-L$ (these points may possibly be an extremity of the interval). Let $y \in L^2((-1, 0), \mathbb{C})$ be defined by

$$y(t) = \sum_{n=0}^{K} \overline{\gamma_n} \chi_{(t_n, t_n + \delta)}(t). \tag{5.58}$$

We claim that y satisfies (5.57). Consider first the case $t \in (1+L-T, 0)$, in which we have $f(t) = t - L$ since $(1+L-T, 0) \subset [L-1, 0)$. Since $f(1+L-T) = 1-T$ and $t_0 = -1$, it follows by construction of δ that $f(t) \notin (t_0, t_0+\delta)$. If $t \notin \bigcup_{n=0}^{K}(t_n, t_n+\delta)$, then $f(t) \notin \bigcup_{n=0}^{K}(t_n, t_n+\delta)$; indeed, $f(t) \in (t_n, t_n+\delta)$ for some $n \in [\![1, K]\!]$ implies immediately, by construction of f and δ, that $t \in (t_{n-1}, t_{n-1}+\delta)$. Hence, if $t \in (1+L-T, 0) \setminus \bigcup_{n=0}^{K}(t_n, t_n+\delta)$, one immediately has that $y(t) = y(t-L) = 0$ and hence the second equation of (5.57) is satisfied for such t. Notice that $f(t_K) = t_{K+1} < 1-T$, so that $t_K < 1+L-T$, and thus, by construction of δ, $(t_K, t_K+\delta) \cap (1+L-T, 0) = \emptyset$. If $t \in (t_n, t_n+\delta)$ for some $n \in [\![0, K-1]\!]$, one has $t_n \in (1+L-T, 0) \subset [L-1, 0)$ by construction of δ and $f(t) \in (t_{n+1}, t_{n+1}+\delta)$, which shows, by the construction of $(\gamma_n)_{n=0}^{K}$, that

$$\overline{\alpha} y(t) + \overline{\beta} y(t-L) = \overline{\alpha \gamma_n} + \overline{\beta} \overline{\gamma_{n+1}} = 0.$$

Hence the second equation of (5.57) is satisfied for every $t \in (1+L-T, 0)$.

Consider now the case $t \in (-1, L-1)$, in which we have $f(t) = t+1-L$. Since $f^{-1}(t_0, t_0+\delta) = (L-1, L-1+\delta)$, one has $f(t) \notin (t_0, t_0+\delta)$. Again, the same argument as before shows that, if $t \notin \bigcup_{n=0}^{K}(t_n, t_n+\delta)$, then $f(t) \notin \bigcup_{n=0}^{K}(t_n, t_n+\delta)$, and thus, for such t, $y(t) = y(t+1-L) = 0$ and the first equation of (5.57) is satisfied. Since $f(t_K) = t_{K+1} \in [-1, 1-T)$, one has $t_K \in [L-1, 1+L-T)$, and hence $(t_K, t_K+\delta) \cap (-1, L-1) = \emptyset$. If $t \in (t_n, t_n+\delta) \cap (-1, L-1)$ for some $n \in [\![0, K-1]\!]$, one has $t_n \in (-1, L-1)$ and $f(t) \in (t_{n+1}, t_{n+1}+\delta)$, which shows, by the construction of $(\gamma_n)_{n=0}^{K}$, that

$$\overline{\beta} y(t+1-L) + y(t) = \overline{\beta} \overline{\gamma_{n+1}} + \overline{\gamma_n} = 0.$$

Hence the first equation of (5.57) is satisfied for every $t \in (-1, L-1)$. Thus $E(T)^*$ is not injective, yielding that approximate controllability does not hold. ■

Remark 5.55. The construction of the function $x = (x_1, x_2)$ in the kernel of $E(T)^*$ carried out in the previous proof for the case $1+L \leq T < 2$ can be interpreted in terms of the graphical representation for $E(T)^*$ from Remark 5.37. Notice first that, thanks to (5.53), the only line segments $\sigma_{\mathbf{n}}$ from Remark 5.37 lying inside the domain $[0, T) \times [-1, 0)$ and associated with non-zero coefficients are $\sigma_{(0,0)}$, $\sigma_{(0,1)}$, $\sigma_{(1,0)}$, and $\sigma_{(1,1)}$ (see Figure 5.3).

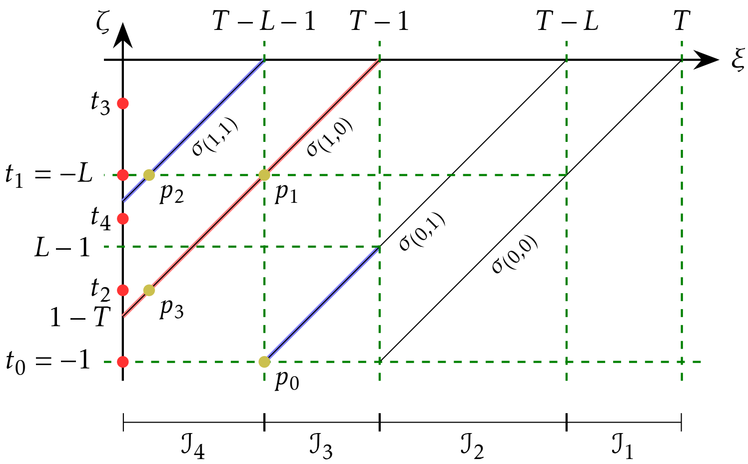


Figure 5.3: Graphical representation for $E(T)^*$ used to construct a nontrivial function x in the kernel of $E(T)^*$ when $1+L \leq T < 2$.

The interval $[0, T)$ can be decomposed in four subintervals $\mathcal{J}_1 = [T-L, T)$, $\mathcal{J}_2 = [T-1, T)$, $\mathcal{J}_3 = [T-L-1, T-1)$, and $\mathcal{J}_4 = [0, T-L-1)$, these subintervals being associated respectively with the four equations of (5.56), thanks to Remark 5.37. Intervals $\mathcal{J}_1$ and $\mathcal{J}_2$ are associated with the first and second equations of (5.56), which are only used to compute x_2 once x_1 is constructed.

The construction of x_1 goes as follows. One wishes x_1 to be equal to $\overline{\gamma_0} = 1$ on a small interval $(t_0, t_0 + \delta)$ from $t_0 = -1$. By looking at the intersection of the horizontal line $\zeta = t_0$ with the line segments $\sigma_{\mathbf{n}}$ inside the region $(\mathcal{I}_3 \cup \mathcal{I}_4) \times [-1, 0)$, one notices that it only intersects the line segment $\sigma_{(0,1)}$, highlighted in blue in Figure 5.3, this intersection happening at the point denoted by p_0 in the figure. From Remark 5.37, the point p_0 corresponds to the term $x_1(t)$ for $t = -1$ in the third equation of (5.56), and hence, for this equation to be satisfied in a small interval $(-1, -1+\delta)$, one needs x_1 to be equal to $\overline{\gamma_1} = -\overline{\gamma_0}/\overline{\beta}$ in the interval $(-L, -L+\delta)$ corresponding to the other term of this equation. This other term can be obtained graphically by finding the intersection of the vertical line passing through p_0 with another of the segments $\sigma_{\mathbf{n}}$, which happens at the point p_1, when the vertical line intersects the segment $\sigma_{(1,0)}$, highlighted in red in the figure. The point p_1 has a ζ-coordinate of $t_1 = -L$, which means that x_1 must be equal to $\overline{\gamma_1}$ in the interval $(-L, -L + \delta)$. However, the horizontal line $\zeta = t_1$ also intersects another segment, $\sigma_{(1,1)}$, at the point p_2, and so one repeats the construction from t_1, until one arrives at a point t_K ($K = 4$ in Figure 5.3) such that the horizontal line $\zeta = t_K$ only intersects one segment $\sigma_{\mathbf{n}}$ in $(\mathcal{I}_3 \cup \mathcal{I}_4) \times [-1, 0)$, which will necessarily be the segment $\sigma_{(1,0)}$. This corresponds to the condition $t_K \in [L-1, L+1-T)$. When x_1 is constructed up to this point, the third and fourth equations of (5.56) are satisfied at all times, and hence x_1 satisfies the required properties.

Notice also that one can modify the previous construction to obtain a smooth function $x \in \mathcal{C}_0^\infty([-1, 0), \mathbb{C}^2)$ in the kernel of $E(T)^*$, simply by replacing the characteristic functions $\chi_{(t_n, t_n+\delta)}$ in (5.58) by $\varphi(\cdot - t_n)$ for a certain $\mathcal{C}^\infty$ function φ compactly supported in $(0, \delta)$.

We are now left to prove Theorem 5.51(c) in the case $T \geq 2\Lambda_1$. The next lemma shows that it suffices to consider the case $T = 2\Lambda_1$.

Lemma 5.56. *Let $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, and $(\Lambda_1, \Lambda_2) \in (0, +\infty)^2$ with $\Lambda_1 > \Lambda_2$, and assume that (A_1, B) and (A_2, B) are controllable. Then the following assertions hold.*

(a) *System (5.41) is exactly controllable in some time $T \geq 2\Lambda_1$ if and only if it is exactly controllable in time $T = 2\Lambda_1$.*

(b) *System (5.41) is approximately controllable in some time $T \geq 2\Lambda_1$ if and only if it is approximately controllable in time $T = 2\Lambda_1$.*

Proof. Thanks to Remark 5.54, it suffices to consider the case where A_1, A_2, B, and (Λ_1, Λ_2) are given by (5.49), in which case $E(T)^*$ is given by (5.55).

It is trivial that exact controllability in $T = 2$ implies exact controllability for larger time. To prove the converse, it suffices to show that, for every $T \geq 2$, there exists $C_T > 0$ such that, for every $x \in \mathsf{X}$,

$$\|E(T)^* x\|_{\mathsf{Y}_T}^2 \leq C_T \|E(2)^* x\|_{\mathsf{Y}_2}^2.$$

Indeed, let $T \geq 2$, $x = (x_1, x_2) \in \mathsf{X}$. Since the right-hand side of (5.55) does not depend on T, one obtains that, for $t \in (-2, 0)$, $(E(T)^* x)(t + T) = (E(2)^* x)(t + 2)$. Hence

$$\begin{aligned}
\|E(T)^* x\|_{\mathsf{Y}_T}^2 &= \int_0^T |(E(T)^* x)(t)|^2 \, dt = \int_{-T}^0 |(E(T)^* x)(t + T)|^2 \, dt \\
&= \|E(2)^* x\|_{\mathsf{Y}_2}^2 + \int_{-T}^{-2} |(E(T)^* x)(t + T)|^2 \, dt \\
&= \|E(2)^* x\|_{\mathsf{Y}_2}^2 + \int_{-T}^{-2} \left| \overline{\alpha}^{-\lfloor t \rfloor - 2} \overline{\beta} x_1(\{t\} - 1) + \overline{\alpha}^{-\lfloor t+L \rfloor - 1} x_1(\{t + L\} - 1) \right|^2 dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \|E(2)^* x\|_{\mathsf{Y}_2}^2 + \sum_{k=1}^{\lceil T\rceil-2} \int_{-(k+2)}^{-(k+1)} \left|\overline{\alpha}^{-\lfloor t\rfloor-2}\overline{\beta} x_1(\{t\}-1) + \overline{\alpha}^{-\lfloor t+L\rfloor-1} x_1(\{t+L\}-1)\right|^2 dt \\
&= \|E(2)^* x\|_{\mathsf{Y}_2}^2 + \sum_{k=1}^{\lceil T\rceil-2} |\alpha|^k \int_{-2}^{-1} \left|\overline{\alpha}^{-\lfloor t\rfloor-2}\overline{\beta} x_1(\{t\}-1) + \overline{\alpha}^{-\lfloor t+L\rfloor-1} x_1(\{t+L\}-1)\right|^2 dt \\
&\leq \|E(2)^* x\|_{\mathsf{Y}_2}^2 \sum_{k=0}^{\lceil T\rceil-2} |\alpha|^k,
\end{aligned}
$$

and one can thus conclude the proof by taking $C_T = \sum_{k=0}^{\lceil T\rceil-2} |\alpha|^k$.

Concering approximate controllability, it is also trivial that approximate controllability in $T = 2$ implies approximate controllability for larger time. Suppose now that the system is approximately controllable in time $T \geq 2$ and take $x \in \mathsf{X}$ such that $E(2)^* x = 0$ in Y_2. Thanks to (5.55), this means that, for almost every $t \in (-2, 0)$,

$$
\begin{cases}
x_2(t) = 0, & \text{if } -L < t < 0, \\
x_2(t) + x_1(t+L) = 0, & \text{if } -1 < t < -L, \\
\overline{\alpha}^{-\lfloor t\rfloor-2}\overline{\beta} x_1(\{t\}-1) + \overline{\alpha}^{-\lfloor t+L\rfloor-1} x_1(\{t+L\}-1) = 0, & \text{if } -2 < t < -1.
\end{cases}
$$

Multiplying the last equation by $\overline{\alpha}^k$ for $k \in \mathbb{N}^*$ shows that, for almost every $t \in (-\infty, 0)$,

$$
\begin{cases}
x_2(t) = 0, & \text{if } -L < t < 0, \\
x_2(t) + x_1(t+L) = 0, & \text{if } -1 < t < -L, \\
\overline{\alpha}^{-\lfloor t\rfloor-2}\overline{\beta} x_1(\{t\}-1) + \overline{\alpha}^{-\lfloor t+L\rfloor-1} x_1(\{t+L\}-1) = 0, & \text{if } t < -1.
\end{cases}
$$

In particular, $E(T)^* x = 0$ in Y_T, and thus $x = 0$ in X, which shows the approximate controllability in time 2. ■

In order to prove Theorem 5.51(c) in the case $T = 2\Lambda_1$, we introduce some notation.

Definition 5.57. Let $\alpha, \beta \in \mathbb{C}$ and $p, q \in \mathbb{N}^*$ with p, q coprime. We define the bounded linear operator $S : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ by

$$
Sx(t) = \begin{cases}
\overline{\beta} x(t) + x(t+L-1) & \text{if } -L < t < 0, \\
\overline{\beta} x(t) + \overline{\alpha} x(t+L) & \text{if } -1 < t < -L,
\end{cases} \tag{5.59}
$$

and the matrix $M = (m_{ij})_{i,j \in [\![1,q]\!]} \in \mathcal{M}_q(\mathbb{C})$ by

$$
m_{ij} = \begin{cases}
\overline{\beta}, & \text{if } j = i, \\
\overline{\alpha}, & \text{if } j = i - p, \\
1, & \text{if } j = i + q - p, \\
0, & \text{otherwise}.
\end{cases} \tag{5.60}
$$

By a straightforward computation, one obtains that the adjoint operator $S^* : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ is given, for $x \in L^2((-1,0),\mathbb{C})$, by

$$
S^* x(t) = \begin{cases}
\beta x(t) + \alpha x(t-L), & \text{if } L-1 < t < 0, \\
\beta x(t) + x(t-L+1), & \text{if } -1 < t < L-1.
\end{cases} \tag{5.61}
$$

The operator S allows one to characterize exact and approximate controllability for (5.41), as shown in the next lemma.

Lemma 5.58. *Let $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, $B \in \mathcal{M}_{2,1}(\mathbb{C})$, and $(\Lambda_1, \Lambda_2) \in (0, +\infty)^2$ with $\Lambda_1 > \Lambda_2$, and assume that (A_1, B) and (A_2, B) are controllable. Then the following assertions hold.*

(a) *System (5.41) is exactly controllable in some time $T \geq 2\Lambda_1$ if and only if S^* is surjective or, equivalently, if there exists $c > 0$ such that $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq c\|x\|_{L^2((-1,0),\mathbb{C})}$ for every $x \in L^2((-1,0),\mathbb{C})$.*

(b) *System (5.41) is approximately controllable in some time $T \geq 2\Lambda_1$ if and only if S is injective.*

Proof. Thanks to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.49), in which case $E(T)$ and $E(T)^*$ are given respectively by (5.54) and (5.55).

Let us first prove (b). Combining Lemma 5.56 and Proposition 5.35, one obtains that (5.41) is approximately controllable in some time $T \geq 2$ if and only if $E(2)^*$ is injective. Thanks to (5.55) and (5.59), $x = (x_1, x_2) \in \mathsf{X}$ satisfies $E(2)^* x = 0$ if and only if

$$\begin{cases} x_2(t) = 0, & \text{if } -L < t < 0, \\ x_2(t) = -x_1(t+L), & \text{if } -1 < t < -L, \\ Sx_1(t) = 0, & \text{if } -1 < t < 0. \end{cases} \tag{5.62}$$

Assume that $E(2)^*$ is injective and let $w \in L^2((-1,0),\mathbb{C})$ be such that $Sw = 0$. Defining $x = (x_1, x_2) \in \mathsf{X}$ by $x_1 = w$, $x_2(t) = 0$ for $t \in (-L, 0)$, and $x_2(t) = -w(t+L)$ for $t \in (-1, -L)$, one obtains from (5.62) that $E(2)^* x = 0$, which implies that $x = 0$ and hence $w = 0$, yielding the injectivity of S. Assume now that S is injective and let $x = (x_1, x_2) \in \mathsf{X}$ be such that $E(2)^* x = 0$. Then, by the third equation of (5.62), one has $Sx_1 = 0$, which shows that $x_1 = 0$, and thus the first two equations of (5.62) show that $x_2 = 0$, yielding the injectivity of $E(2)^*$. Hence the injectivity of $E(2)^*$ is equivalent to that of S.

Let us now prove (a). Combining Lemma 5.56 and Proposition 5.34, one obtains that (5.41) is exactly controllable in some time $T \geq 2$ if and only if $E(2)$ is surjective. Thanks to (5.54), one has, for $u \in \mathsf{Y}_2$,

$$(E(2)u)(t) = \begin{cases} \begin{pmatrix} \beta u(t+1) + \alpha u(t+1-L) + u(t+2-L) \\ u(t+2) \end{pmatrix}, & \text{if } L-1 < t < 0, \\ \begin{pmatrix} \beta u(t+1) + u(t+2-L) \\ u(t+2) \end{pmatrix}, & \text{if } -1 < t < L-1. \end{cases} \tag{5.63}$$

Assume that $E(2)$ is surjective and take $w \in L^2((-1,0),\mathbb{C})$. Let $x = (w, 0) \in \mathsf{X}$ and take $u \in \mathsf{Y}_2$ such that $E(2)u = x$. Hence, by (5.63), one has that $u(t+2) = 0$ for $t \in (-1,0)$, i.e., $u(t) = 0$ for $t \in (1,2)$. Hence $u(t+2-L) = 0$ for $L-1 < t < 0$, and one obtains from (5.63) that

$$\begin{cases} \beta u(t+1) + \alpha u(t+1-L) = w(t), & \text{if } L-1 < t < 0, \\ \beta u(t+1) + u(t+2-L) = w(t), & \text{if } -1 < t < L-1. \end{cases}$$

This shows that $S^* u(\cdot + 1) = w$, and thus S^* is surjective. Assume now that S^* is surjective and take $x = (x_1, x_2) \in \mathsf{X}$. Let $\widetilde{u} \in L^2((-1,0),\mathbb{C})$ be such that

$$S^* \widetilde{u}(t) = \begin{cases} x_1(t) - x_2(t-L), & \text{if } L-1 < t < 0, \\ x_1(t), & \text{if } -1 < t < L-1, \end{cases} \tag{5.64}$$

and define $u \in \mathsf{Y}_2$ by $u(t) = \widetilde{u}(t-1)$ if $0 < t < 1$ and $u(t) = x_2(t-2)$ if $1 < t < 2$. Then, combining (5.61), (5.63), and (5.64), one obtains that $E(2)u = x$, which yields the surjectivity of $E(2)$. Hence the surjectivity of $E(2)$ is equivalent to that of S^*. The fact that the latter is equivalent to the existence of $c > 0$ such that $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq c\|x\|_{L^2((-1,0),\mathbb{C})}$ for every $x \in L^2((-1,0),\mathbb{C})$ is a classical result in functional analysis (see, e.g., [153, Theorem 4.13]). ∎

Remark 5.59. As in Remark 5.37, one can provide a graphical representation for the operators S and S^*. Notice first that, as in Remark 5.55, for A_1, A_2, B, and (Λ_1, Λ_2) under the form (5.49), the only line segments $\sigma_{\mathbf{n}}$ from Remark 5.37 lying inside the domain $[0,2)\times[-1,0)$ and associated with non-zero coefficients are $\sigma_{(0,0)}$, $\sigma_{(0,1)}$, $\sigma_{(1,0)}$, and $\sigma_{(1,1)}$, which are associated respectively with the coefficients $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}\beta\\0\end{pmatrix}$, and $\begin{pmatrix}\alpha\\0\end{pmatrix}$.

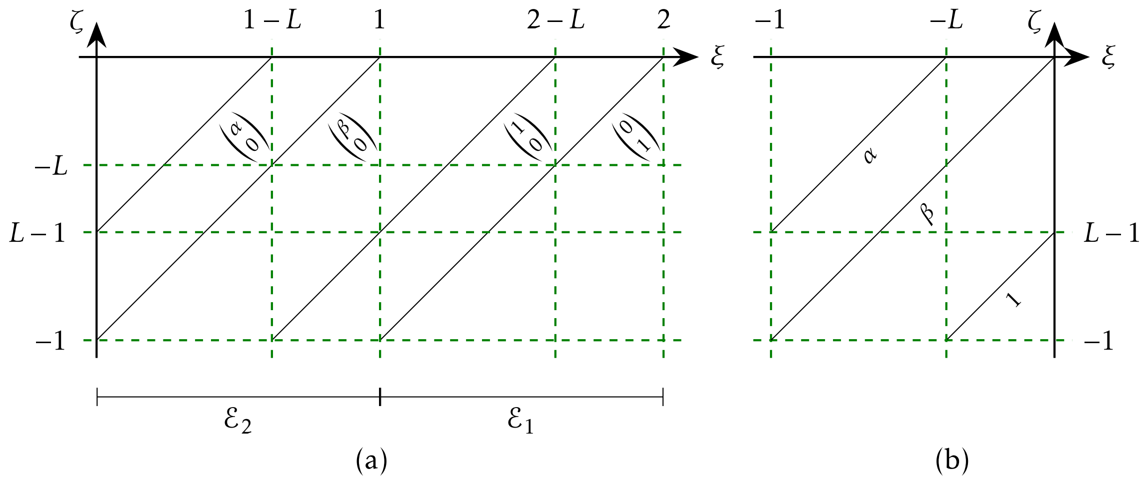


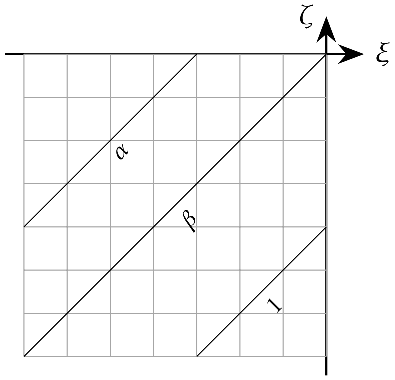
Figure 5.4: Graphical representations of the operators (a) $E(2)$ and $E(2)^*$, and (b) S and S^*.

Figure 5.4(a) provides the graphical representation for $E(2)$ and $E(2)^*$ given in Remark 5.37. One can decompose the domain $[0,2)\times[-1,0)$ in two parts, $\mathcal{E}_1 = [1,2)\times[-1,0)$ and $\mathcal{E}_2 = [0,1)\times[-1,0)$. The value of $E(2)^*x(t)$ for $t \in [0,1)$, which corresponds to the region $\mathcal{E}_2$, only depends on x_1, and S is defined as the operator that, to each x_1, associates the value of $E(2)^*x(t)$ for $t \in (0,1)$, translated by 1 in order to obtain as a result a function defined in $(-1,0)$. Hence S can be seen as the part of $E(2)^*$ corresponding to the region $\mathcal{E}_2$, which is represented in Figure 5.4(b). It turns out that this part of $E(2)^*$ is enough to characterize its injectivity and the surjectivity of its adjoint, as shown in Lemma 5.58. The intuition behind this fact is that, if, for instance, one is interested in studying the injectivity of $E(2)^*$, by looking for non-trivial solutions $x = (x_1, x_2) \in \mathsf{X}$ of $E(2)^*x = 0$, the part of $E(2)^*$ in $\mathcal{E}_1$, corresponding to $E(2)^*x(t)$ for $t \in [1,2)$, provides the equations $x_2(t) = 0$ for $t \in (-L,0)$ and $x_2(t) = -x_1(t+L)$ for $t \in (-1,-L)$, thanks to (5.55). These equations completely characterize x_2 once x_1 is computed, and hence looking for non-trivial solutions of $E(2)^*x = 0$ amounts to considering only the part of this equation corresponding to $\mathcal{E}_2$ and the variable x_1. A similar argument holds for the surjectivity of $E(2)$.

The matrix M defined by (5.60) corresponds to a representation of S when $L = \frac{p}{q}$, similar to the construction of C and E from $E(T)$ performed in Remark 5.48. Indeed, by decomposing $(-1,0)^2$ into squares $S_{ij} = \left(-\frac{i}{q}, -\frac{i-1}{q}\right)\times\left(-\frac{j}{q}, -\frac{j-1}{q}\right)$ for $i, j \in [\![1,q]\!]$, one remarks that the intersection between one of the line segments representing S and the square S_{ij} is either empty or equal to the diagonal of the square from its bottom left corner to its top right corner, the coefficient M_{ij} being zero in the first case or the conjugate of the coefficient corresponding to the intersecting line in the second case. Figure 5.5 provides this construction in the case $L = \frac{3}{7}$.

We next gather some properties of the matrix M defined in (5.60).

Lemma 5.60. *The characteristic polynomial of M is $P(\lambda) = \left(\lambda - \overline{\beta}\right)^q - \overline{\alpha}^{q-p}$.*



(a)

$$M = \begin{pmatrix} \overline{\beta} & 0 & 0 & \overline{\alpha} & 0 & 0 & 0 \\ 0 & \overline{\beta} & 0 & 0 & \overline{\alpha} & 0 & 0 \\ 0 & 0 & \overline{\beta} & 0 & 0 & \overline{\alpha} & 0 \\ 0 & 0 & 0 & \overline{\beta} & 0 & 0 & \overline{\alpha} \\ 1 & 0 & 0 & 0 & \overline{\beta} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \overline{\beta} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \overline{\beta} \end{pmatrix}$$

(b)

Figure 5.5: Construction of the matrix M from S in the case $L = \frac{3}{7}$.

Proof. We have $P(\lambda) = \det(\lambda \operatorname{Id}_q - M)$. Setting $M_\lambda = \lambda \operatorname{Id}_q - M$ and writing $M_\lambda = (m_{ij}^{(\lambda)})_{i,j \in [\![1,q]\!]}$, we have

$$m_{ij}^{(\lambda)} = \begin{cases} \lambda - \overline{\beta}, & \text{if } j = i, \\ -\overline{\alpha}, & \text{if } j = i - p, \\ -1, & \text{if } j = i + q - p, \\ 0, & \text{otherwise.} \end{cases} \tag{5.65}$$

Let $\mathfrak{S}_q$ denote the group of permutations of $[\![1,q]\!]$ and $\epsilon(\sigma)$ denote the signature of an element $\sigma \in \mathfrak{S}_q$. Leibniz formula for the determinant gives

$$P(\lambda) = \det M_\lambda = \sum_{\sigma \in \mathfrak{S}_q} \epsilon(\sigma) \prod_{i=1}^{q} m_{i\sigma(i)}^{(\lambda)}. \tag{5.66}$$

Thanks to (5.65), the product $\prod_{i=1}^{q} m_{i\sigma(i)}^{(\lambda)}$ is nonzero only if $\sigma \in \mathfrak{S}_q$ satisfies, for every $i \in [\![1,q]\!]$,

$$\sigma(i) \in \begin{cases} \{i, i + q - p\}, & \text{if } i \in [\![1,p]\!], \\ \{i, i - p\}, & \text{if } i \in [\![p+1,q]\!]. \end{cases} \tag{5.67}$$

Let $\tau \in \mathfrak{S}_q$ be the translation by -1 modulo q, i.e., $\tau(i) = i - 1$ if $i \in [\![2,q]\!]$ and $\tau(1) = q$. We have $\epsilon(\tau) = (-1)^{q-1}$, and thus $\epsilon(\tau^p) = (-1)^{(q-1)p}$. Since p,q are coprime, one has $pq \equiv p + q + 1 \mod 2$ and thus $(q-1)p \equiv q + 1 \mod 2$, which gives $\epsilon(\tau^p) = (-1)^{q+1}$. Notice, moreover, that (5.67) can be written as $\sigma(i) \in \{i, \tau^p(i)\}$ for every $i \in [\![1,q]\!]$.

We claim that the only permutations $\sigma \in \mathfrak{S}_q$ satisfying (5.67) are $\operatorname{Id}_{\mathfrak{S}_q}$ and τ^p. Indeed, it is clear that these two permutations satisfy (5.67). Suppose now that σ is a permutation satisfying (5.67) such that $\sigma \neq \operatorname{Id}_{\mathfrak{S}_q}$. Hence there exists $i_0 \in [\![1,q]\!]$ such that $\sigma(i_0) \neq i_0$, and thus, by (5.67), $\sigma(i_0) = \tau^p(i_0)$. We claim that $\sigma(\tau^{kp}(i_0)) = \tau^{(k+1)p}(i_0)$ for every $k \in \mathbb{N}$. Indeed, this holds for $k = 0$, and, if $k \in \mathbb{N}$ is such that $\sigma(\tau^{kp}(i_0)) = \tau^{(k+1)p}(i_0)$, then, since $\tau^{(k+1)p}(i_0) \neq \tau^{kp}(i_0)$, one has $\sigma(\tau^{(k+1)p}(i_0)) \neq \sigma(\tau^{kp}(i_0)) = \tau^{(k+1)p}(i_0)$, which implies by (5.67) that $\sigma(\tau^{(k+1)p}(i_0)) = \tau^p(\tau^{(k+1)p}(i_0)) = \tau^{(k+2)p}(i_0)$, which concludes the proof by induction that $\sigma(\tau^{kp}(i_0)) = \tau^{(k+1)p}(i_0)$ for every $k \in \mathbb{N}$. Now, since p,q are coprime, $\{\tau^{kp}(i_0) \mid k \in \mathbb{N}\} = [\![1,q]\!]$, and thus $\sigma(i) = \tau^p(i)$ for every $i \in [\![1,q]\!]$, which shows that $\sigma = \tau^p$.

It now follows from (5.66) that

$$P(\lambda) = \prod_{i=1}^{q} m_{ii}^{(\lambda)} + (-1)^{q+1} \prod_{i=1}^{q} m_{i\tau^p(i)}^{(\lambda)} = \left(\lambda - \overline{\beta}\right)^q + (-1)^{q+1}(-1)^q \overline{\alpha}^{q-p} = \left(\lambda - \overline{\beta}\right)^q - \overline{\alpha}^{q-p},$$

which concludes the proof. ∎

Corollary 5.61. *The determinant of M is* $\det M = \overline{\beta}^q - (-1)^q \overline{\alpha}^{q-p}$.

Proof. Letting P be the characteristic polynomial of M, one has from Lemma 5.60 that $\det M = (-1)^q \det(-M) = (-1)^q P(0) = \overline{\beta}^q - (-1)^q \overline{\alpha}^{q-p}$. ∎

We can now prove Theorem 5.51(c)(ii) in the case $T \geq 2\Lambda_1$.

Proof of Theorem 5.51(c)(ii). Assume that (A_1, B) and (A_2, B) are controllable, in which case, according to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.49). Since one has already proved that approximate controllability does not hold for $T < 2$, it suffices to show that, for $T \geq 2$, the system is approximately controllable if and only if $0 \notin C$. Thanks to Lemma 5.58, one is left to show that the operator S defined in (5.59) is injective if and only if $0 \notin C$. We write in this proof $\alpha = |\alpha| e^{i\theta}$ for some $\theta \in (-\pi, \pi]$.

Consider first the case $L \in (0,1) \cap \mathbb{Q}$ and write $L = \frac{p}{q}$ for $p, q \in \mathbb{N}^*$ coprime. Define the operator $R : L^2((-1,0), \mathbb{C}) \to L^2((-1/q, 0), \mathbb{C}^q)$ by

$$(Rx(t))_n = x\left(t - \frac{n-1}{q}\right), \qquad -\frac{1}{q} < t < 0,\ n \in [\![1, q]\!].$$

One immediately verifies from its definition that R is a unitary transformation and that, for every $x \in L^2((-1/q, 0), \mathbb{C}^q)$,

$$(RSR^{-1}x)(t) = Mx(t), \tag{5.68}$$

where M is the matrix defined in (5.60). One has

$$C = \left\{ \beta + |\alpha|^{1-\frac{p}{q}} e^{i(\theta + 2k\pi)\left(1 - \frac{p}{q}\right)} \;\middle|\; k \in [\![0, q-1]\!] \right\}. \tag{5.69}$$

Notice that $0 \in C$ if and only if $\det M = 0$. Indeed, from Corollary 5.61, one has $\det M = 0$ if and only if $(-\beta)^q = \alpha^{q-p}$, i.e., if and only if $-\beta$ is a q-th root of α^{q-p}, which means that $-\beta = |\alpha|^{\frac{q-p}{q}} e^{i(\theta+2k\pi)\frac{q-p}{q}}$ for some $k \in [\![0, q-1]\!]$, this being equivalent to $0 \in C$. Since R is a unitary transformation, one obtains in particular that the injectivity of S is equivalent to that of RSR^{-1}, which, thanks to (5.68), is equivalent to that of M. Since M is injective if and only if $\det M \neq 0$, one concludes that S is injective if and only if $0 \notin C$, as required.

Assume now that $L \in (0,1) \setminus \mathbb{Q}$. Let $x \in L^2((-1,0), \mathbb{C})$ be such that $Sx = 0$. Then

$$x(t) = \begin{cases} -\dfrac{1}{\overline{\beta}} x(t + L - 1), & \text{if } -L < t < 0, \\[2ex] -\dfrac{\overline{\alpha}}{\overline{\beta}} x(t + L), & \text{if } -1 < t < -L. \end{cases}$$

Let $\varphi : [-1, 0) \to [-1, 0)$ be defined by $\varphi(t) = t + L$ if $t \in [-1, -L)$ and $\varphi(t) = t + L - 1$ if $t \in [-L, 0)$. The function φ is a translation by L modulo 1 on the interval $[-1, 0)$, and can also be seen as an interval exchange transformation. Since L is irrational, φ is ergodic with respect to the Lebesgue measure in $[-1, 0)$ (see, e.g., [122, Chapter II, Theorem 3.2]). We have

$$x(t) = -\frac{\overline{\alpha} \chi_{(-1,-L)}(t) + \chi_{(-L,0)}(t)}{\overline{\beta}} x \circ \varphi(t) \qquad \text{for } -1 < t < 0.$$

Choose $\gamma \in \mathbb{C}$ such that $e^{\gamma(1-L)} = -\overline{\beta}$; when $0 \in C$, we impose further that γ satisfies $e^{\gamma} = \overline{\alpha}$. This choice is possible in the latter case since the condition $0 \in C$ means that there exists $k \in \mathbb{Z}$ such that $\beta + |\alpha|^{1-L} e^{i(\theta+2k\pi)(1-L)} = 0$, which implies in particular that $\alpha \neq 0$ since $\beta \neq 0$ by its definition. Hence it suffices to take $\gamma = \log|\alpha| - i(\theta + 2k\pi)$ and one immediately has $e^{\gamma} = \overline{\alpha}$ and $e^{\gamma(1-L)} = -\overline{\beta}$. Let $y \in L^2((-1,0),\mathbb{C})$ be defined by $y(t) = e^{\gamma t}x(t)$. Then y satisfies

$$y(t) = \left(\overline{\alpha}e^{-\gamma}\chi_{(-1,-L)}(t) + \chi_{(-L,0)}(t)\right) y \circ \varphi(t) \qquad \text{for } -1 < t < 0. \tag{5.70}$$

If $0 \in C$, then $\overline{\alpha}e^{-\gamma} = 1$, and thus y satisfies $y = y \circ \varphi$. Since φ is ergodic with respect to the Lebesgue measure in $[-1,0)$, the set of functions $y \in L^2((-1,0),\mathbb{C})$ satisfying $y = y \circ \varphi$ is the set of functions constant almost everywhere (see, e.g., [122, Chapter II, Proposition 2.1]). Hence the solutions of $Sx = 0$ are the functions of the form $x(t) = Ce^{-\gamma t}$ for $C \in \mathbb{C}$, which shows that S is not injective and hence (5.41) is not approximately controllable in time $T \geq 2$.

If $0 \notin C$, notice that, from (5.70),

$$\|y\|^2_{L^2((-1,0),\mathbb{C})} = |\overline{\alpha}e^{-\gamma}|^2 \int_{L-1}^{0} \left|y(t)\right|^2 dt + \int_{-1}^{L-1} \left|y(t)\right|^2 dt,$$

which shows that

$$\left(1 - |\overline{\alpha}e^{-\gamma}|^2\right) \int_{L-1}^{0} \left|y(t)\right|^2 dt = 0.$$

If $|\overline{\alpha}e^{-\gamma}| \neq 1$, then y is zero in the interval $(L-1,0)$. By (5.70), it follows that y is zero in $\varphi^{-k}(L-1,0)$ for every $k \in \mathbb{N}$, which shows that $y = 0$ in $(-1,0)$ since φ is ergodic (see, e.g., [172, Theorem 1.5]). Hence $x = 0$ is the unique solution of $Sx = 0$, and thus (5.41) is approximately controllable in time $T \geq 2$. If $|\overline{\alpha}e^{-\gamma}| = 1$, write $\overline{\alpha}e^{-\gamma} = e^{i\frac{2\pi\eta L}{1-L}}$ for some $\eta \in \left[0, \frac{1-L}{L}\right)$. Notice that, for every $n \in \mathbb{Z}$, one has $e^{i\frac{2\pi(\eta-n)L}{1-L}} \neq 1$; indeed, one has $\overline{\alpha} = e^{\gamma + i\frac{2\pi\eta L}{1-L}}$ and hence the possible complex values of $\overline{\alpha}^{1-L}$ are

$$\overline{\alpha}^{1-L} = e^{\gamma(1-L)+i(2\pi\eta L+2\pi k(1-L))} = -\overline{\beta}e^{i2\pi L(\eta-k)}, \qquad k \in \mathbb{Z}. \tag{5.71}$$

If $e^{i\frac{2\pi(\eta-n)L}{1-L}} = 1$ for some $n \in \mathbb{Z}$, then $\eta \equiv n \mod \frac{1-L}{L}$ and, since $\frac{1-L}{L} = \frac{1}{L} - 1$, we conclude that there exists $k \in \mathbb{Z}$ such that $\eta \equiv k \mod \frac{1}{L}$. For such k, $e^{i2\pi L(\eta-k)} = 1$, which is not possible due to (5.71) since we are in the case $0 \notin C$. Hence, for every $n \in \mathbb{Z}$, one has $e^{i\frac{2\pi(\eta-n)L}{1-L}} \neq 1$. The function y satisfies

$$y(t) = \left(e^{i\frac{2\pi\eta L}{1-L}}\chi_{(-1,-L)}(t) + \chi_{(-L,0)}(t)\right) y \circ \varphi(t) \qquad \text{for } -1 < t < 0.$$

Thus, for every $n \in \mathbb{Z}$,

$$\begin{aligned}\int_{-1}^{0} y(t)e^{i\frac{2\pi n}{1-L}t}dt &= e^{i\frac{2\pi\eta L}{1-L}} \int_{-1}^{-L} y(t+L)e^{i\frac{2\pi n}{1-L}t}dt + \int_{-L}^{0} y(t+L-1)e^{i\frac{2\pi n}{1-L}t}dt \\ &= e^{i\frac{2\pi(\eta-n)L}{1-L}} \int_{L-1}^{0} y(t)e^{i\frac{2\pi n}{1-L}t}dt + \int_{-1}^{L-1} y(t)e^{i\frac{2\pi n}{1-L}t}dt,\end{aligned}$$

which implies that

$$\left(1 - e^{i\frac{2\pi(\eta-n)L}{1-L}}\right) \int_{L-1}^{0} y(t)e^{i\frac{2\pi n}{1-L}t}dt = 0, \qquad \forall n \in \mathbb{Z}.$$

Since $e^{i\frac{2\pi(\eta-n)L}{1-L}} \neq 1$ for every $n \in \mathbb{Z}$, we conclude that

$$\int_{L-1}^{0} y(t)e^{i\frac{2\pi n}{1-L}t}dt = 0, \qquad \forall n \in \mathbb{Z},$$

which shows that all the Fourier coefficients of $y|_{(L-1,0)}$ vanish. Thus y is zero in the interval $(L-1,0)$ and, as before, this implies that y is zero in the interval $(-1,0)$. Hence S is injective, and thus (5.41) is approximately controllable in time $T \geq 2$. ∎

Remark 5.62. One can also obtain from the previous proof that, if $L = \frac{p}{q}$ for some $p, q \in \mathbb{N}^*$ coprime, then approximate and exact controllability in time $T \geq 2$ are equivalent for (5.41). Indeed, notice that, when (5.41) is approximately controllable in time $T \geq 2$, then $0 \notin C$, M is invertible, and hence, by (5.68), one has $\left\|RSR^{-1}x\right\|_{L^2((-1/q,0),\mathbb{C}^q)} \geq \left|M^{-1}\right|_2^{-1} \|x\|_{L^2((-1/q,0),\mathbb{C}^q)}$ for every $x \in L^2((-1/q,0),\mathbb{C}^q)$, which shows that $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq \left|M^{-1}\right|_2^{-1} \|x\|_{L^2((-1,0),\mathbb{C})}$ for every $x \in L^2((-1,0),\mathbb{C})$, thus giving the exact controllability of (5.41) in time $T \geq 2$ thanks to Lemma 5.58. This agrees with the general result of Proposition 5.47 for commensurable delays. Moreover, one obtains from (5.69) that the set C is finite, which shows that $\overline{C} = C$ and hence conditions $0 \notin C$ and $0 \notin \overline{C}$ are equivalent. This proves Theorem 5.51(c)(i) in the case where Λ_1 and Λ_2 are commensurable, i.e., $\frac{\Lambda_2}{\Lambda_1} \in \mathbb{Q}$.

Remark 5.63. When $0 \in C$ and $L \notin \mathbb{Q}$, this proof also shows that the kernel of S is the vector space spanned by the function $x(t) = e^{\gamma t}$ with $\gamma \in \mathbb{C}$ chosen as in the proof of the theorem. Thanks to (5.55), this means that the kernel of $E(2)^*$ is the vector space spanned by the function

$$x(t) = \begin{pmatrix} e^{-\gamma t} \\ -e^{-\gamma(t+L)}\chi_{(-1,-L)}(t) \end{pmatrix}.$$

Remark 5.64. When $0 \in C$, $L \notin \mathbb{Q}$, and $\alpha, \beta \in \mathbb{R}$, one can always choose $\gamma \in \mathbb{R}$, obtaining thus a real-valued nonzero solution to $Sx = 0$, and hence to $E(2)^* x = 0$. Indeed, notice first that one can only have $0 \in C$ with $\alpha, \beta \in \mathbb{R}$ if $\alpha > 0$ (in which case $\beta < 0$), since $\alpha = 0$ implies $\beta = 0$, which is not possible, and, for $\alpha < 0$, the equality $\beta + \alpha^{1-L} = 0$ for some complex value of α^{1-L} implies that $-\beta = \alpha^{1-L} = |\alpha|^{1-L} e^{i(\pi + 2n\pi)(1-L)}$ for some $n \in \mathbb{Z}$, but such expression cannot be real for any $n \in \mathbb{Z}$ since $L \notin \mathbb{Q}$. Now, when $\alpha > 0$, it suffices to take $\gamma = \log \alpha \in \mathbb{R}$ and the conditions required for γ in the proof are satisfied.

In order to complete the proof of Theorem 5.51, one only needs to show part (c)(i) for $T \geq 2\Lambda_1$. Before doing so, let us provide some more properties of the matrix M defined in (5.60).

Lemma 5.65. *Let $p, q \in \mathbb{N}^*$ be coprime with $p < q$ and set $r = q - p$. Assume that $\alpha \neq 0$ and write $\alpha = |\alpha| e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. The eigenvalues of the matrix M defined in (5.60) are*

$$\lambda_j = \overline{\beta} + |\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}} e^{i\frac{2\pi j r}{q}}, \qquad j \in [\![1, q]\!]. \tag{5.72}$$

For $j \in [\![1, q]\!]$, a right eigenvector $v_j \in \mathbb{C}^q \simeq \mathcal{M}_{q,1}(\mathbb{C})$ of M associated with λ_j is

$$v_j = \left(|\alpha|^{\frac{k}{q}} e^{-i\frac{\theta k}{q}} e^{i\frac{2\pi j k}{q}}\right)_{k=1}^{q}$$

and a left eigenvector $w_j \in \mathcal{M}_{1,q}(\mathbb{C})$ of M associated with λ_j is

$$w_j = \frac{1}{q}\left(|\alpha|^{-\frac{k}{q}} e^{i\frac{\theta k}{q}} e^{-i\frac{2\pi j k}{q}}\right)_{k=1}^{q}.$$

Moreover, for every $j, k \in [\![1, q]\!]$, we have $w_k v_j = \delta_{jk}$.

Proof. Formula (5.72) for the eigenvalues of M follows immediately from the expression of the characteristic polynomial of M given in Lemma 5.60.

Let $j \in [\![1, q]\!]$. For $k \in [\![1, p]\!]$,

$$\begin{aligned}
(Mv_j)_k &= \overline{\beta} |\alpha|^{\frac{k}{q}} e^{-i\frac{\theta k}{q}} e^{i\frac{2\pi j k}{q}} + |\alpha|^{\frac{k+q-p}{q}} e^{-i\frac{\theta(k+q-p)}{q}} e^{i\frac{2\pi j(k+q-p)}{q}} \\
&= |\alpha|^{\frac{k}{q}} e^{-i\frac{\theta k}{q}} e^{i\frac{2\pi j k}{q}} \left(\overline{\beta} + |\alpha|^{\frac{q-p}{q}} e^{-i\frac{\theta(q-p)}{q}} e^{i\frac{2\pi j(q-p)}{q}}\right) = \lambda_j (v_j)_k,
\end{aligned}$$

and, for $k \in [\![p+1, q]\!]$,

$$\begin{aligned}(Mv_j)_k &= \overline{\beta}\,|\alpha|^{\frac{k}{q}} e^{-i\frac{\theta k}{q}} e^{i\frac{2\pi jk}{q}} + \overline{\alpha}\,|\alpha|^{\frac{k-p}{q}} e^{-i\frac{\theta(k-p)}{q}} e^{i\frac{2\pi j(k-p)}{q}} \\ &= |\alpha|^{\frac{k}{q}} e^{-i\frac{\theta k}{q}} e^{i\frac{2\pi jk}{q}} \left(\overline{\beta} + |\alpha|^{\frac{q-p}{q}} e^{-i\frac{\theta(q-p)}{q}} e^{i\frac{2\pi j(q-p)}{q}}\right) = \lambda_j (v_j)_k,\end{aligned}$$

which shows that $Mv_j = \lambda_j v_j$, and hence v_j is a right eigenvector of M associated with λ_j. Now, for $k \in [\![1, q-p]\!]$,

$$\begin{aligned}(w_j M)_k &= \frac{1}{q}\overline{\beta}\,|\alpha|^{-\frac{k}{q}} e^{i\frac{\theta k}{q}} e^{-i\frac{2\pi jk}{q}} + \frac{1}{q}\overline{\alpha}\,|\alpha|^{-\frac{k+p}{q}} e^{i\frac{\theta(k+p)}{q}} e^{-i\frac{2\pi j(k+p)}{q}} \\ &= \frac{1}{q}|\alpha|^{-\frac{k}{q}} e^{i\frac{\theta k}{q}} e^{-i\frac{2\pi jk}{q}} \left(\overline{\beta} + |\alpha|^{\frac{q-p}{q}} e^{-i\frac{\theta(q-p)}{q}} e^{i\frac{2\pi j(q-p)}{q}}\right) = \lambda_j (w_j)_k,\end{aligned}$$

and, for $k \in [\![q-p+1, q]\!]$,

$$\begin{aligned}(w_j M)_k &= \frac{1}{q}\overline{\beta}\,|\alpha|^{-\frac{k}{q}} e^{i\frac{\theta k}{q}} e^{-i\frac{2\pi jk}{q}} + \frac{1}{q}|\alpha|^{-\frac{k+p-q}{q}} e^{i\frac{\theta(k+p-q)}{q}} e^{-i\frac{2\pi j(k+p-q)}{q}} \\ &= \frac{1}{q}|\alpha|^{-\frac{k}{q}} e^{i\frac{\theta k}{q}} e^{-i\frac{2\pi jk}{q}} \left(\overline{\beta} + |\alpha|^{\frac{q-p}{q}} e^{-i\frac{\theta(q-p)}{q}} e^{i\frac{2\pi j(q-p)}{q}}\right) = \lambda_j (w_j)_k,\end{aligned}$$

which shows that $w_j M = \lambda_j w_j$, and hence w_j is a left eigenvector of M associated with λ_j. Finally, for $j, k \in [\![1, q]\!]$, one evaluates immediately $w_k v_j = \frac{1}{q}\sum_{\ell=1}^{q} e^{i\frac{2\pi(j-k)\ell}{q}} = \delta_{jk}$. ∎

Let $V, W, D \in \mathcal{M}_q(\mathbb{C})$ be defined by

$$V = (V_{jk})_{j,k\in[\![1,q]\!]}, \qquad W = (W_{jk})_{j,k\in[\![1,q]\!]}, \qquad D = (D_{jk})_{j,k\in[\![1,q]\!]},$$

with, for $j, k \in [\![1, q]\!]$

$$V_{jk} = (v_k)_j, \qquad W_{jk} = (w_j)_k, \qquad D_{jk} = \lambda_j \delta_{jk}.$$

It follows from classical results from linear algebra (and also from straightforward computations from Lemma 5.65) that

$$M = VDW \qquad \text{and} \qquad V = W^{-1}.$$

Hence, if M is invertible, then $M^{-1} = VD^{-1}W$. One can now provide the following upper bound on the norm of the inverse of M.

Lemma 5.66. *Let $p, q \in \mathbb{N}^*$ be coprime with $p < q$, set $r = q - p$, and let M be given by (5.60). If $\alpha \neq 0$ and $\left|\beta\right| \neq |\alpha|^{\frac{r}{q}}$, then M is invertible and*

$$\left|M^{-1}\right|_2 \leq \frac{\max\left(|\alpha|, |\alpha|^{-1}\right)}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|}.$$

Proof. By Corollary 5.61, M is invertible if and only if $\overline{\beta}^q - (-1)^q \overline{\alpha}^r \neq 0$, and thus M is invertible when $\alpha \neq 0$ and $\left|\beta\right| \neq |\alpha|^{\frac{r}{q}}$. In this case, $M^{-1} = VD^{-1}W$ and thus, for $j, k \in [\![1, q]\!]$,

$$\begin{aligned}\left(M^{-1}\right)_{jk} &= \sum_{\ell=1}^{q} (v_\ell)_j \lambda_\ell^{-1} (w_\ell)_k = \frac{|\alpha|^{\frac{j-k}{q}} e^{-i\theta\frac{j-k}{q}}}{q} \sum_{\ell=1}^{q} \lambda_\ell^{-1} e^{i\frac{2\pi\ell(j-k)}{q}} = \\ &= \frac{|\alpha|^{\frac{j-k}{q}} e^{-i\theta\frac{j-k}{q}}}{q} \sum_{\ell=1}^{q} \frac{e^{i\frac{2\pi\ell(j-k)}{q}}}{\overline{\beta} + |\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}} e^{i\frac{2\pi\ell r}{q}}} = \frac{|\alpha|^{\frac{j-k}{q}} e^{-i\theta\frac{j-k}{q}}}{q\overline{\beta}} \sum_{\ell=1}^{q} \frac{e^{i\frac{2\pi\ell(j-k)}{q}}}{1 + \frac{|\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}}}{\overline{\beta}} e^{i\frac{2\pi\ell r}{q}}}.\end{aligned} \tag{5.73}$$

We claim that, for every $z \in \mathbb{C}$ such that $z^q \neq 1$, we have

$$\sum_{\ell=1}^{q} \frac{e^{i\frac{2\pi\ell(j-k)}{q}}}{1 - ze^{i\frac{2\pi\ell r}{q}}} = \frac{qz^{d_{j,k}}}{1-z^q}, \tag{5.74}$$

where $d_{j,k}$ is the unique integer in $[\![0, q-1]\!]$ such that $rd_{j,k} + j - k \equiv 0 \mod q$, which is well-defined since q and r are coprime.

To show that (5.74) holds for every $z \in \mathbb{C}$ such that $z^q \neq 1$, it suffices to show that it holds for $z \in \mathbb{C}$ with $|z| < 1$, since both left- and right-hand sides of (5.74) are meromorphic functions with simple poles at the q roots of $z^q = 1$. If $z \in \mathbb{C}$ is such that $|z| < 1$, then

$$\sum_{\ell=1}^{q} \frac{e^{i\frac{2\pi\ell(j-k)}{q}}}{1 - ze^{i\frac{2\pi\ell r}{q}}} = \sum_{\ell=1}^{q} e^{i\frac{2\pi\ell(j-k)}{q}} \sum_{s=0}^{\infty} z^s e^{i\frac{2\pi\ell rs}{q}} = \sum_{s=0}^{\infty} z^s \sum_{\ell=1}^{q} e^{i\frac{2\pi\ell(rs+j-k)}{q}} = qz^{d_{j,k}} \sum_{t=0}^{\infty} z^{tq} = \frac{qz^{d_{j,k}}}{1-z^q},$$

where we use that $\sum_{\ell=1}^{q} e^{i\frac{2\pi\ell(rs+j-k)}{q}} = q$ if $rs + j - k \equiv 0 \mod q$ and is equal to zero otherwise, and that $\{s \in \mathbb{N} \mid rs + j - k \equiv 0 \mod q\} = \{d_{j,k} + tq \mid t \in \mathbb{N}\}$. Hence (5.74) is proved.

Since $\left|\beta\right| \neq |\alpha|^{\frac{r}{q}}$ implies $\overline{\beta}^q \neq (-1)^q \overline{\alpha}^r$, we have $\left(-\frac{|\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}}}{\overline{\beta}}\right)^q \neq 1$. Hence, combining (5.73) and (5.74), we obtain that

$$\left(M^{-1}\right)_{jk} = \frac{|\alpha|^{\frac{j-k}{q}} e^{-i\theta\frac{j-k}{q}}}{q\overline{\beta}} \frac{q\left(-\frac{|\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}}}{\overline{\beta}}\right)^{d_{j,k}}}{1 - \left(-\frac{|\alpha|^{\frac{r}{q}} e^{-i\frac{\theta r}{q}}}{\overline{\beta}}\right)^q} = (-1)^{d_{j,k}} \frac{\overline{\alpha}^{n_{j,k}} \overline{\beta}^{q-1-d_{j,k}}}{\overline{\beta}^q - (-1)^q \overline{\alpha}^r},$$

where $n_{j,k} \in \mathbb{Z}$ is the unique integer satisfying $rd_{j,k} + j - k = n_{j,k} q$; moreover, since $d_{j,k} \in [\![0, q-1]\!]$ and $j, k \in [\![1, q]\!]$, we have $n_{j,k} \in [\![0, r]\!]$.

Notice that, for $j, k \in [\![1, q]\!]$, $\frac{rd_{j,k}}{q} = n_{j,k} + \frac{k-j}{q}$, and hence $n_{j,k} = \left\lfloor \frac{rd_{j,k}}{q} \right\rfloor + \delta_{j>k}$, where $\delta_{j>k} = 1$ if $j > k$ and $\delta_{j>k} = 0$ otherwise. Thus, for $k \in [\![1, q]\!]$,

$$\sum_{j=1}^{q} \left|\left(M^{-1}\right)_{jk}\right| = \frac{1}{\left|\beta^q - (-1)^q \alpha^r\right|} \sum_{j=1}^{q} |\alpha|^{\left\lfloor \frac{rd_{j,k}}{q} \right\rfloor + \delta_{j>k}} \left|\beta\right|^{q-1-d_{j,k}}.$$

Since $d_{j,k}$ is defined as the unique integer in $[\![0, q-1]\!]$ satisfying $rd_{j,k} + j - k \equiv 0 \mod q$ and r, q are coprime, we obtain that, for fixed $k \in [\![1, q]\!]$, the map $j \mapsto d_{j,k}$ is a bijection between $[\![1, q]\!]$ and $[\![0, q-1]\!]$. Hence, when $|\alpha| \geq 1$,

$$\begin{aligned}
\sum_{j=1}^{q} \left|\left(M^{-1}\right)_{jk}\right| &\leq \frac{|\alpha|}{\left|\beta^q - (-1)^q \alpha^r\right|} \sum_{j=0}^{q-1} |\alpha|^{\left\lfloor \frac{rj}{q} \right\rfloor} \left|\beta\right|^{q-1-j} \leq \frac{|\alpha| \left|\beta\right|^{q-1}}{\left|\beta^q - (-1)^q \alpha^r\right|} \sum_{j=0}^{q-1} |\alpha|^{\frac{rj}{q}} \left|\beta\right|^{-j} \\
&= \frac{|\alpha| \left|\beta\right|^{q-1}}{\left|\beta^q - (-1)^q \alpha^r\right|} \left|\frac{1 - |\alpha|^r \left|\beta\right|^{-q}}{1 - |\alpha|^{\frac{r}{q}} \left|\beta\right|^{-1}}\right| = \frac{|\alpha|}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|} \frac{\left|\left|\beta\right|^q - |\alpha|^r\right|}{\left|\beta^q - (-1)^q \alpha^r\right|} \leq \frac{|\alpha|}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|},
\end{aligned}$$

and, similarly, when $0 < |\alpha| < 1$,

$$\begin{aligned}
\sum_{j=1}^{q} \left|\left(M^{-1}\right)_{jk}\right| &\leq \frac{1}{\left|\beta^q - (-1)^q \alpha^r\right|} \sum_{j=0}^{q-1} |\alpha|^{\left\lfloor \frac{rj}{q} \right\rfloor} \left|\beta\right|^{q-1-j} \leq \frac{|\alpha|^{-1} \left|\beta\right|^{q-1}}{\left|\beta^q - (-1)^q \alpha^r\right|} \sum_{j=0}^{q-1} |\alpha|^{\frac{rj}{q}} \left|\beta\right|^{-j} \\
&\leq \frac{|\alpha|^{-1}}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|},
\end{aligned}$$

which shows that

$$\left|M^{-1}\right|_1 = \max_{k\in[\![1,q]\!]} \sum_{j=1}^{q} \left|\left(M^{-1}\right)_{jk}\right| \leq \frac{\max\left(|\alpha|, |\alpha|^{-1}\right)}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|}.$$

A similar argument also shows that

$$\left|M^{-1}\right|_\infty = \max_{j\in[\![1,q]\!]} \sum_{k=1}^{q} \left|\left(M^{-1}\right)_{jk}\right| \leq \frac{\max\left(|\alpha|, |\alpha|^{-1}\right)}{\left|\left|\beta\right| - |\alpha|^{\frac{r}{q}}\right|},$$

and the result follows since $\left|M^{-1}\right|_2 \leq \sqrt{\left|M^{-1}\right|_1 \left|M^{-1}\right|_\infty}$. ■

Lemma 5.66 allows one to finally conclude the proof of Theorem 5.51.

Proof of Theorem 5.51(c)(i). Assume that (A_1, B) and (A_2, B) are controllable, in which case, according to Remark 5.54, we can assume that A_1, A_2, B, and (Λ_1, Λ_2) are under the form (5.49). Since one has already proved that exact controllability does not hold for $T < 2$, it suffices to show that, for $T \geq 2$, the system is exactly controllable if and only if $0 \notin \overline{C}$. Remark 5.62 has already shown the result when $L \in (0,1) \cap \mathbb{Q}$, and thus one is left to prove only the case $L \in (0,1) \setminus \mathbb{Q}$. Thanks to Lemma 5.58, one is left to show that $0 \notin \overline{C}$ if and only if S^* is surjective or, equivalently, if there exists $c > 0$ such that the operator S defined in (5.59) satisfies $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq c\|x\|_{L^2((-1,0),\mathbb{C})}$ for every $x \in L^2((-1,0),\mathbb{C})$. We write in this proof $\alpha = |\alpha| e^{i\theta}$ for some $\theta \in (-\pi, \pi]$.

Take $L \in (0,1) \setminus \mathbb{Q}$. Notice first that $0 \in \overline{C}$ if and only if $\left|\beta\right| = |\alpha|^{1-L}$. Indeed, one has

$$C = \left\{\beta + |\alpha|^{1-L} e^{i(\theta+2k\pi)(1-L)} \;\middle|\; k \in \mathbb{Z}\right\},$$

and, since L is irrational, one immediately computes that $\overline{C}$ is the circle in $\mathbb{C}$ of center β and radius $|\alpha|^{1-L}$. Hence $0 \in \overline{C}$ if and only if $\left|\beta\right| = |\alpha|^{1-L}$.

Let us first treat the case $\alpha = 0$. Since $\beta \neq 0$, one has $0 \notin \overline{C}$ in this case. We will prove the exact controllability of (5.41) by showing the surjectivity of S^*. Take $x \in L^2((-1,0),\mathbb{C})$ and define $u \in L^2((-1,0),\mathbb{C})$ by

$$u(t) = \sum_{k=0}^{\left\lfloor \frac{t}{L-1} \right\rfloor} \frac{(-1)^k}{\beta^{k+1}} x(t + k(1-L)).$$

Then, for $L - 1 < t < 0$, one has $S^* u(t) = \beta u(t) = x(t)$ and, for $-1 < t < L - 1$, one has

$$\begin{aligned}
S^* u(t) &= \beta u(t) + u(t - L + 1) \\
&= \sum_{k=0}^{\left\lfloor \frac{t}{L-1} \right\rfloor} \frac{(-1)^k}{\beta^k} x(t + k(1-L)) + \sum_{k=0}^{\left\lfloor \frac{t-L+1}{L-1} \right\rfloor} \frac{(-1)^k}{\beta^{k+1}} x(t - L + 1 + k(1-L)) \\
&= \sum_{k=0}^{\left\lfloor \frac{t}{L-1} \right\rfloor} \frac{(-1)^k}{\beta^k} x(t + k(1-L)) + \sum_{k=1}^{\left\lfloor \frac{t}{L-1} \right\rfloor} \frac{(-1)^{k-1}}{\beta^k} x(t + k(1-L)) = x(t),
\end{aligned}$$

which shows that $S^* u = x$ and thus S^* is surjective.

Consider now the case $\alpha \neq 0$. Suppose that $0 \notin \overline{C}$, which means that $\left|\beta\right| \neq |\alpha|^{1-L}$. Let $(p_n), (q_n)$ be two sequences of positive integers such that p_n and q_n are coprime for every $n \in \mathbb{N}$ and $\frac{p_n}{q_n} \to L$ as $n \to \infty$. Let $r_n = q_n - p_n$. Up to eliminating a finite number of terms in the sequence, we can assume that $\left|\beta\right| \neq |\alpha|^{\frac{r_n}{q_n}}$ for every $n \in \mathbb{N}$. Let $S_n : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ be given by

$$S_n x(t) = \begin{cases} \overline{\beta}x(t) + x\left(t + \frac{p_n}{q_n} - 1\right) & \text{if } -\frac{p_n}{q_n} < t < 0, \\ \overline{\beta}x(t) + \overline{\alpha}x\left(t + \frac{p_n}{q_n}\right) & \text{if } -1 < t < -\frac{p_n}{q_n}, \end{cases}$$

and $S : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ be defined by (5.59). One easily verifies (using, e.g., [152, Theorem 9.5]) that, for every $x \in L^2((-1,0),\mathbb{C})$, one has $S_n x \to Sx$ as $n \to \infty$. Since $\left|\beta\right| \neq |\alpha|^{\frac{r_n}{q_n}}$ for every $n \in \mathbb{N}$, we obtain, from Remark 5.62, that

$$\|S_n x\|_{L^2((-1,0),\mathbb{C})} \geq \left|M_n^{-1}\right|_2^{-1} \|x\|_{L^2((-1,0),\mathbb{C})},$$

where M_n is given by (5.60) with p and q replaced respectively by p_n and q_n. Hence, by Lemma 5.66,

$$\|S_n x\|_{L^2((-1,0),\mathbb{C})} \geq \frac{\left|\left|\beta\right| - |\alpha|^{\frac{r_n}{q_n}}\right|}{\max\left(|\alpha|, |\alpha|^{-1}\right)} \|x\|_{L^2((-1,0),\mathbb{C})},$$

and, letting $n \to \infty$,

$$\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq \frac{\left|\left|\beta\right| - |\alpha|^{1-L}\right|}{\max\left(|\alpha|, |\alpha|^{-1}\right)} \|x\|_{L^2((-1,0),\mathbb{C})},$$

which proves that (5.41) is exactly controllable in time $T \geq 2$.

Suppose now that $0 \in \overline{C}$, i.e., that $\left|\beta\right| = |\alpha|^{1-L}$. For $a, b \in \mathbb{C}$, let $S_{a,b} : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ be defined by

$$S_{a,b} x(t) = \begin{cases} \overline{b}x(t) + x(t + L - 1) & \text{if } -L < t < 0, \\ \overline{b}x(t) + \overline{a}x(t + L) & \text{if } -1 < t < -L. \end{cases}$$

In particular, for every $\lambda \in \mathbb{C}$, one has $S_{a,b} - \lambda = S_{a,b-\overline{\lambda}}$. Let $\sigma_{\mathrm{p}}(S_{a,b})$ denote the set of eigenvalues of $S_{a,b}$. Thus $\lambda \in \sigma_{\mathrm{p}}(S_{a,b})$ if and only if $0 \in \sigma_{\mathrm{p}}(S_{a,b-\overline{\lambda}})$, which, by the proof of Theorem 5.51(c)(ii), is the case if and only if $\overline{b} - \lambda + \overline{a}^{1-L} = 0$ for some complex value of $\overline{a}^{1-L}$. Hence $\sigma_{\mathrm{p}}(S)$ is the set of all possible values of $\overline{\beta} + \overline{\alpha}^{1-L}$, and, since L is irrational and thanks to the condition $\left|\beta\right| = |\alpha|^{1-L}$, we conclude that $0 \in \overline{\sigma_{\mathrm{p}}(S)}$. Hence there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\sigma_{\mathrm{p}}(S)$ such that $\lambda_n \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, let x_n be an eigenfunction of S associated with the eigenvalue λ_n and with $\|x_n\|_{L^2((-1,0),\mathbb{C})} = 1$. Hence $Sx_n = \lambda_n x_n \to 0$ as $n \to +\infty$, which shows that there does not exist $c > 0$ such that $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq c\|x\|_{L^2((-1,0),\mathbb{C})}$ for every $x \in L^2((-1,0),\mathbb{C})$, and thus (5.41) is not exactly controllable. ∎

5.A Alternative proof of Theorem 5.27

The proof of Theorem 5.27 relies on the corresponding result for delay vectors Λ with commensurable components from Lemma 5.26 — which is proved using the augmented system from Lemma 5.24 — and on Theorems 5.20 and 5.22, relating the relative controllability of systems with different delays in terms of their rational dependence structure. When

$(1,1,\dots,1) \succcurlyeq \Lambda$, one can give an alternative, more direct proof of Theorem 5.27, through a technique very similar to the one used in Theorem 4.36 to prove a generalization of the Hale–Silkowski criterion to the case of time-varying matrices A_j. We provide below such alternative proof, which also yields Corollary 5.29.

Theorem 5.67. *Let $A = (A_1,\dots,A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, and $\Lambda = (\Lambda_1,\dots,\Lambda_N) \in (0,+\infty)^N$ and suppose that $(1,1,\dots,1) \succcurlyeq \Lambda$. Then $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \lvert\mathbf{n}\rvert_1 \leq d-1,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d. \tag{5.75}$$

Moreover, in such case, its minimal controllability time $T_{\min}$ satisfies $T_{\min} \leq (d-1)\Lambda_{\max}$.

Proof. Let $V(\Lambda)$ be defined as in (4.8). It follows from Proposition 4.9 that there exists $h \in [\![1,N]\!]$ and $M \in \mathcal{M}_{N,h}(\mathbb{N})$ such that $\operatorname{rk} M = h$, $\Lambda = M\ell$ for some $\ell \in (0,+\infty)^h$ with rationally independent components, and $V(\Lambda) = \operatorname{Ran} M$.

Notice that, since $(1,1,\dots,1) \succcurlyeq \Lambda$, for every $\mathbf{n},\mathbf{n}' \in \mathbb{N}^N$ such that $\Lambda\cdot\mathbf{n} = \Lambda\cdot\mathbf{n}'$, one has $\lvert\mathbf{n}\rvert_1 = (1,1,\dots,1)\cdot\mathbf{n} = (1,1,\dots,1)\cdot\mathbf{n}' = \lvert\mathbf{n}'\rvert_1$. We set $\lvert[\mathbf{n}]_\Lambda\rvert_1 = \lvert\mathbf{n}\rvert_1$ for every $\mathbf{n} \in \mathbb{N}^N$. Moreover, for every $\theta = (\theta_1,\dots,\theta_N) \in V(\Lambda)$, if $\mathbf{n},\mathbf{n}' \in \mathbb{N}^N$ are such that $\Lambda\cdot\mathbf{n} = \Lambda\cdot\mathbf{n}'$, then $\theta\cdot\mathbf{n} = \theta\cdot\mathbf{n}'$, and thus we set $\theta\cdot[\mathbf{n}]_\Lambda = \theta\cdot\mathbf{n}$ for every $\mathbf{n} \in \mathbb{N}^N$.

For $k \in \mathbb{N}$, let

$$\mathcal{A}_k = \operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \lvert\mathbf{n}\rvert_1 = k\right\}.$$

Since $\Lambda_{\min}\lvert\mathbf{n}\rvert_1 \leq \Lambda\cdot\mathbf{n} \leq \Lambda_{\max}\lvert\mathbf{n}\rvert_1$ for every $\mathbf{n} \in \mathbb{N}^N$, one has, for every $T > 0$,

$$\begin{aligned}
&\operatorname{Span}\left\{\Xi Bw \;\middle|\; \Xi \in \mathcal{A}_k,\ k \leq \left\lfloor\frac{T}{\Lambda_{\max}}\right\rfloor,\ w \in \mathbb{C}^m\right\}\\
\subset{}&\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\}\\
\subset{}&\operatorname{Span}\left\{\Xi Bw \;\middle|\; \Xi \in \mathcal{A}_k,\ k \leq \left\lfloor\frac{T}{\Lambda_{\min}}\right\rfloor,\ w \in \mathbb{C}^m\right\}.
\end{aligned} \tag{5.76}$$

In particular, $\Sigma(A,B,\Lambda)$ is relatively controllable in some time $T > 0$ if and only if there exists $K \in \mathbb{N}$ such that

$$\operatorname{Span}\{\Xi Bw \mid \Xi \in \mathcal{A}_k,\ k \leq K,\ w \in \mathbb{C}^m\} = \mathbb{C}^d,$$

or, equivalently,

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} Bw \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \lvert\mathbf{n}\rvert_1 \leq K,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d. \tag{5.77}$$

Define $F : V(\Lambda) \to \mathcal{M}_d(\mathbb{C})$ by

$$F(\theta) = \sum_{j=1}^N e^{i\theta_j} A_j. \tag{5.78}$$

We claim that, for $k \in \mathbb{N}$, $\mathcal{A}_k = \operatorname{Span}\left\{F(\theta)^k \;\middle|\; \theta \in V(\Lambda)\right\}$. Indeed, notice that

$$F(\theta)^k = \left(\sum_{j=1}^N e^{i\theta_j} A_j\right)^k = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \lvert\mathbf{n}\rvert_1 = k}} e^{i\theta\cdot\mathbf{n}}\Xi_{\mathbf{n}} = \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \lvert[\mathbf{n}]\rvert_1 = k}} e^{i\theta\cdot[\mathbf{n}]} \sum_{\mathbf{n}'\in[\mathbf{n}]} \Xi_{\mathbf{n}'} = \sum_{\substack{[\mathbf{n}]\in\mathcal{N}_\Lambda\\ \lvert[\mathbf{n}]\rvert_1 = k}} e^{i\theta\cdot[\mathbf{n}]}\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda}.$$

Hence $\mathcal{A}_k \supset \operatorname{Span}\left\{F(\theta)^k \;\middle|\; \theta \in V(\Lambda)\right\}$. Let $P \in \mathcal{M}_h(\mathbb{R})$ be an invertible matrix such that $\ell = Pe_1$, where e_1 denotes the first vector of the canonical basis of $\mathbb{R}^h$. Then $\Lambda = MPe_1$. For $k \in \mathbb{N}$, define $f_k : \mathbb{R}^h \to \mathcal{M}_d(\mathbb{C})$ by

$$f_k(\nu) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N\\ \lvert\mathbf{n}\rvert_1 = k}} e^{i\mathbf{n}\cdot MP\nu}\Xi_{\mathbf{n}}.$$

Hence $f_k(\nu) = F(MP\nu)^k$ for every $\nu \in \mathbb{R}^h$. One has that, for every $\mathbf{n} \in \mathbb{N}^N$, if $k = \lvert[\mathbf{n}]_\Lambda\rvert_1$, then it follows from (4.34) that

$$\lim_{R\to+\infty} \frac{1}{(2R)^h} \int_{[-R,R]^h} f_k(\nu) e^{-i\mathbf{n}\cdot MP\nu}\, d\nu = \sum_{\substack{\mathbf{n}'\in[\mathbf{n}]_\Lambda\\ \lvert\mathbf{n}'\rvert_1 = k}} \Xi_{\mathbf{n}'} = \widehat{\Xi}^{\Lambda}_{[\mathbf{n}]}$$

(notice that the hypothesis $(1,1,\dots,1) \succcurlyeq \Lambda$ is crucial here). Since $\operatorname{Span}\left\{F(\theta)^k \;\middle|\; \theta \in V(\Lambda)\right\}$ is closed and contains $f_k(\nu)$ for every $\nu \in \mathbb{R}^h$, it follows that $\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} \in \operatorname{Span}\left\{F(\theta)^k \;\middle|\; \theta \in V(\Lambda)\right\}$ for every $[\mathbf{n}] \in \mathcal{N}_\Lambda$ such that $\lvert[\mathbf{n}]\rvert_1 = k$, which yields $\mathcal{A}_k = \operatorname{Span}\left\{F(\theta)^k \;\middle|\; \theta \in V(\Lambda)\right\}$.

Since $F(\theta) \in \mathcal{M}_d(\mathbb{C})$ for every $\theta \in V(\Lambda)$, it follows from Cayley–Hamilton Theorem that $\mathcal{A}_k \subset \bigcup_{j=0}^{d-1} \mathcal{A}_j$ for every $k \in \mathbb{N}$. That (5.75) is equivalent to the relative controllability of $\Sigma(A, B, \Lambda)$ in some time $T > 0$ is thus a consequence of the fact that the latter is equivalent to (5.77) for some $K \in \mathbb{N}$.

In order to obtain the bound on the minimal controllability time from the statement, assume that $\Sigma(A, B, \Lambda)$ is relatively controllable in time T for some $T > 0$. Then, by (5.76), one has

$$\operatorname{Span}\left\{\Xi B w \;\middle|\; \Xi \in \mathcal{A}_k,\ k \leq \left\lfloor \frac{T}{\Lambda_{\min}} \right\rfloor,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d,$$

and, since $\bigcup_{k=0}^{\lfloor T/\Lambda_{\min}\rfloor} \mathcal{A}_k \subset \bigcup_{k=0}^{d-1} \mathcal{A}_k$, one has

$$\operatorname{Span}\left\{\Xi B w \;\middle|\; \Xi \in \bigcup_{k=0}^{d-1} \mathcal{A}_k,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d,$$

which, by (5.76), shows that $\operatorname{Span}\left\{\widehat{\Xi}^{\Lambda}_{[\mathbf{n}]} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda\cdot\mathbf{n} \leq (d-1)\Lambda_{\max},\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d$, and thus $\Sigma(A, B, \Lambda)$ is relatively controllable in time $(d-1)\Lambda_{\max}$, which proves that $T_{\min} \leq (d-1)\Lambda_{\max}$. ■

Annexe A

Résumé des résultats de la thèse

A.1 Introduction

A.1.1 Systèmes à commutation

Au cours des dernières décennies, plusieurs travaux de recherche se sont intéressés à des systèmes dont le comportement peut être décrit par des variables continues et discrètes en interaction, les *systèmes hybrides* [12, 61, 75, 115, 121, 158]. À cause de leurs nombreuses applications, par exemple dans le contrôle de systèmes mécaniques ou de processus industriels, l'industrie automobile, les systèmes électriques de puissance, le contrôle du trafic aérien, les processus chimiques, ou encore les systèmes de transport, les systèmes hybrides ont attiré l'attention des chercheurs non seulement en mathématiques mais aussi dans d'autres domaines, tels que les sciences de l'ingénieur ou l'informatique.

Les *systèmes à commutation* correspondent au point de vue sur les systèmes hybrides où l'intérêt central est la dynamique continue, la variable discrète étant vue comme des modes ou des sous-systèmes qui déterminent l'évolution de la variable continue. Ses applications sont à l'origine du très grand intérêt de recherche qui leur a été porté récemment [113, 114, 123, 158, 166]. Mathématiquement, un système à commutation dans $\mathbb{R}^d$ peut s'écrire sous la forme

$$\dot{x}(t) = f_{\alpha(t)}(x(t)), \qquad t \in \mathbb{R}_+, \tag{A.1}$$

où $x(t)$ est un vecteur dans $\mathbb{R}^d$ ou, plus généralement, dans une variété M ou dans un espace de Banach X, f_k est un champ de vecteurs pour tout k dans un certain ensemble d'indices $\mathcal{I}$, et $\alpha : \mathbb{R}_+ \to \mathcal{I}$ est une fonction constante par morceaux (avec un nombre fini de discontinuités sur tout intervalle borné), appelée *signal de commutation*. En général, α n'est pas complètement connu, l'objectif étant donc d'obtenir des propriétés de (A.1) qui soient robustes par rapport à une certaine classe $\mathcal{G}$ de signaux à commutation. Il est aussi important en pratique de considérer des systèmes de contrôle à commutation, du type

$$\dot{x}(t) = f_{\alpha(t)}(x(t), u(t)), \qquad t \in \mathbb{R}_+, \tag{A.2}$$

où $u(t) \in \mathbb{R}^m$ est une entrée de contrôle.

L'une des principales caractéristiques des systèmes à commutation est le fait que ses propriétés peuvent être assez différentes de celles des sous-systèmes isolés $\dot{x}(t) = f_k(x(t))$. Par exemple, il est possible que des systèmes à commutation composés de sous-systèmes asymptotiquement stables puissent avoir des trajectoires instables et, inversement, des systèmes instables peuvent parfois être stabilisés par un choix approprié de signal à commutation.

Même si la théorie des systèmes à commutation s'est considérablement développée, plusieurs questions importantes sur leur comportement demeurent ouvertes, même dans le cas linéaire. Cela est particulièrement le cas pour les systèmes avec signaux de commutation aléatoire et pour les systèmes à commutation en dimension infinie, qui ont été l'objet de plusieurs travaux de recherche récents [7,17,27,28,76,79,90,111,118,149,169]. Cette thèse présente, dans ses Chapitres 2, 3 et 4, de nouveaux résultats sur la stabilité de systèmes à commutation linéaires, en dimension infinie avec des signaux de commutation déterministes dans les Chapitres 3 et 4, et en dimension finie avec des signaux de commutation aléatoires dans le Chapitre 2, où l'on s'intéresse également au problème de la stabilisation de systèmes de contrôle à commutation.

A.1.2 Systèmes à excitation persistante

Une classe importante de systèmes à commutation, dont l'étude a été la motivation principale pour cette thèse, est celle des *systèmes à excitation persistante*. Il s'agit de systèmes sous la forme (A.2) où le signal de commutation n'affecte que le terme de contrôle, en l'activant ou le désactivant. Dans le cadre linéaire, ce type de système s'écrit

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \qquad t \in \mathbb{R}_+, \tag{A.3}$$

où $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, et $\alpha : \mathbb{R}_+ \to \{0,1\}$, ou $\alpha : \mathbb{R}_+ \to [0,1]$ si l'on autorise des niveaux d'activation intermédiaires pour u.

Plusieurs phénomènes peuvent être représentés par le signal α, tels que des problèmes de transmission de l'actionneur au système, entrainant une action intermittente de la commande u ; des paramètres dépendants du temps qui affectent l'efficacité de la commande u, entrainant l'application effective d'une commande $\alpha(t)u(t)$; l'allocation des ressources de contrôle, choisissant d'agir sur le système uniquement dans des certaines fenêtres de temps ou jusqu'à une certaine valeur de la commande ; parmi d'autres situations possibles. Ces modèles sont particulièrement utiles dans les systèmes contrôlés par des réseaux [93,101,102].

On fait l'hypothèse que la seule information connue sur α est qu'il appartient à une certaine classe de signaux de commutation $\mathcal{G} \subset L^\infty(\mathbb{R},[0,1])$. Pour avoir un problème intéressant du point de vue de la théorie du contrôle, il est important que la classe $\mathcal{G}$ garantisse une action suffisante du contrôle u sur le système. Une façon courante de le faire (voir, par exemple, [46,49,116,126,128,135,164,165]) est de supposer que la classe $\mathcal{G}$ est une classe de signaux à *excitation persistante*.

Définition A.1. Soient $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$. On dit que la fonction $\alpha \in L^\infty(\mathbb{R},[0,1])$ est un *signal à excitation persistante* (T,μ) si, pour tout $t \in \mathbb{R}$, on a

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{A.4}$$

L'ensemble de tous les signaux à excitation persistante (T,μ) est noté par $\mathcal{G}(T,\mu)$. La famille de systèmes

$$\dot{x}(t) = Ax(t) + \alpha(t)Bu(t), \qquad \alpha \in \mathcal{G}(T,\mu), \tag{A.5}$$

est appelé un *système à excitation persistante*.

La condition d'excitation persistante (A.4) provient de problèmes d'identification et de contrôle adaptatif [9–11,37,135], dans lesquels la stabilité asymptotique de l'erreur d'identification de certains paramètres est équivalente, sous certaines hypothèses de régularité, à une condition similaire à (A.4). Néanmoins, l'intérêt de l'étude des systèmes à excitation persistante va bien au delà de ces problèmes, puisque plusieurs autres modèles issus de situations pratiques peuvent s'écrire sous la forme (A.5) ou une généralisation de celle-ci, comme décrit dans [116].

Une partie importante de la littérature sur les systèmes à excitation persistante s'intéresse au problème de leur stabilisation par des retours d'état linéaires $u(t) = Kx(t)$ [39, 45, 49, 128]. Il s'agit de savoir si, étant données $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, et $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$, il existe une matrice K qui rend le système

$$\dot{x}(t) = (A + \alpha(t)BK)x(t) \tag{A.6}$$

exponentiellement stable, uniformément par rapport à $\alpha \in \mathcal{G}(T, \mu)$. Le résultat le plus important dans ce sens est le suivant, montré dans [49]. Rappelons qu'une paire de matrices $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ est dite *stabilisable* s'il existe $K \in \mathcal{M}_{m,d}(\mathbb{R})$ tel que $\dot{x}(t) = (A + BK)x(t)$ soit asymptotiquement stable, et *contrôlable* si la *matrice de contrôlabilité* $\mathcal{C}(A, B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{d-1}B \end{pmatrix} \in \mathcal{M}_{d,dm}(\mathbb{R})$ a rang plein.

Théorème A.2 [49, Théorème 3.2]**.** *Soient $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, et $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$. Supposons que la paire (A, B) soit stabilisable et que les valeurs propres de A aient toutes partie réelle négative ou nulle. Alors il existe $K \in \mathcal{M}_{m,d}(\mathbb{R})$ et des constantes $C, \gamma > 0$ telles que, pour tout $\alpha \in \mathcal{G}(T, \mu)$ et toute solution x de (A.6), on a*

$$|x(t)| \leq Ce^{-\gamma t}|x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

Le Théorème A.2 a été montré d'abord dans le cas des paires contrôlables (A, B) à une entrée, $m = 1$, dans [49, Théorème 3.2], le cas général des systèmes à plusieurs entrées avec la paire (A, B) stabilisable pouvant être obtenu par la décomposition de Kalman et par une récurrence sur le nombre d'entrées (voir, par exemple, [46, Théorème 2.9] et [126, Lemme B.1]). Dans le cas particulier où le système non-contrôlé $\dot{x}(t) = Ax(t)$ est déjà stable (pas nécessairement asymptotiquement), la matrice K peut être choisie indépendante de T et μ [39]. Le résultat du Théorème A.2 a été généralisé dans [126] au cas d'un retour d'état avec retard $u(t) = Kx(t - \tau(t))$. D'autres travaux se sont également intéressés à des retours d'état sous des formes plus générales [151, 164, 165, 173].

L'hypothèse spectrale faite sur A dans l'énoncé du Théorème A.2 n'est pas nécessaire pour la stabilisation de systèmes linéaires du type $\dot{x}(t) = Ax(t) + Bu(t)$. Il a été démontré dans [49, Propositions 4.4 et 4.5] qu'elle n'est pas nécessaire non plus dans le Théorème A.2 si le rapport μ/T est suffisamment grand, mais, si μ/T est petit, il existe des paires stabilisables (A, B), où A admet au moins une valeur propre à partie réelle strictement positive, telles que le système (A.5) ne peut pas être stabilisé asymptotiquement par des retours d'état linéaires.

Une autre question intéressante qui a été traitée dans la littérature est celle de la stabilisation de (A.5) à taux de convergence arbitraire. Il s'agit de savoir si, étant donné $\gamma > 0$, on peut choisir K de tel sorte que le retour d'état $u(t) = Kx(t)$ rende (A.5) exponentiellement stable, ses solutions convergeant vers zéro au moins aussi vite que $e^{-\gamma t}$. Pour les systèmes linéaires du type $\dot{x}(t) = Ax(t) + Bu(t)$, la stabilisation à taux arbitraire est équivalente à la contrôlabilité de la paire (A, B), ce qui est une conséquence du Théorème de placement de pôles (voir, par exemple, [163, Théorème 13]). Dans le cas des systèmes à excitation persistante, la réponse à cette question a été donnée dans [49, Propositions 4.4 et 4.5], où l'on montre que la stabilisation à taux de convergence arbitraire dépend du rapport μ/T.

Proposition A.3 [49, Propositions 4.4 et 4.5]**.**

(a) *Soit $d \in \mathbb{N}^*$. Il existe $\rho^\star \in (0, 1)$, ne dépendant que de d, tel que, pour tous $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$ et $\mu/T > \rho^\star$, toute paire $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,1}(\mathbb{R})$ contrôlable, et tout $\gamma > 0$, il existe $K \in \mathcal{M}_{1,d}(\mathbb{R})$ et $C > 0$ tels que, pour tout $\alpha \in \mathcal{G}(T, \mu)$, toute solution x de (A.6) satisfait*

$$|x(t)| \leq Ce^{-\gamma t}|x(0)|, \qquad \forall t \in \mathbb{R}_+.$$

(b) *Il existe $\rho_\star \in (0,1)$ tel que, pour tous $T, \mu \in \mathbb{R}_+^*$ avec $\mu/T < \rho_\star$ et $(A,B) \in \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_{2,1}(\mathbb{R})$ contrôlable, il existe $\gamma > 0$ tel que, pour tout $K \in \mathcal{M}_{1,2}(\mathbb{R})$, il existe $\alpha \in \mathcal{G}(T,\mu)$ et une solution x de (A.6) pour laquelle $t \mapsto e^{\gamma t}\lvert x(t)\rvert$ n'est pas borné sur $\mathbb{R}_+$.*

La démonstration de la Proposition A.3(b) dans [49] construit, pour chaque matrice $K \in \mathcal{M}_{m,d}(\mathbb{R})$, après une transformation pour se ramener au cas $\gamma = 0$, un signal $\alpha \in \mathcal{G}(T,\mu)$ à valeurs dans $\{0,1\}$ qui déstabilise le système (A.6). Cette construction utilise le phénomène d'*overshoot* qui a lieu lors de la commutation entre les sous-systèmes $\dot{x} = Ax$ et $\dot{x} = (A+BK)x$, correspondant à une possible augmentation de la norme d'une solution d'un système asymptotiquement stable avant qu'elle ne décroisse. Une commutation du sous-système $\dot{x} = (A+BK)x$ vers $\dot{x} = Ax$ après la croissance de la norme due à l'*overshoot* mais avant la décroissance due à la stabilité de $\dot{x} = (A+BK)x$ peut ainsi avoir un effet déstabilisant. Les signaux α déstabilisants construits dans cette preuve sont périodiques et oscillent d'autant plus rapidement que la norme de K est grande.

Cette preuve a conduit à la conjecture, formulée dans [49], que, sous des hypothèses supplémentaires empêchant les commutations trop rapides du signal à excitation persistante α, il pourrait être possible d'obtenir des taux de convergence exponentielle arbitraires pour (A.6). Cette conjecture a été démontrée pour des signaux α lipschitziens et des systèmes en dimension 2 avec une borne uniforme sur la constante de Lipschitz dans [128, Théorème 3.1], pour des signaux à valeurs dans $\{0,1\}$ avec une borne uniforme sur leur variation totale sur tout intervalle de temps de longueur T dans [46, Théorème 4.3], et dans le cas où $\operatorname{rk} B = d$ dans [46, Théorème 4.4].

Le fait que les signaux à excitation persistante déstabilisant (A.6) construits dans la preuve de la Proposition A.3(b) dans [49] soient des signaux à commutation assez rapide et à des instants de temps très spécifiques a également conduit à la conjecture que, si l'on considère (A.5) avec des signaux α issus d'un processus aléatoire, sous des hypothèses assez raisonnables sur celui-ci, ce serait possible de retrouver la stabilisation *presque sure* de (A.6) avec taux de convergence exponentielle arbitraire. L'étude de cette conjecture est la motivation principale du Chapitre 2 de cette thèse, qui fait d'abord une étude du comportement asymptotique de systèmes à commutation avec signaux de commutation aléatoires, caractérisant leurs exposants de Lyapunov par le Théorème ergodique multiplicatif d'Oseledets appliqué à un système associé en temps discret, avant de montrer un résultat de stabilisation de systèmes de contrôle à commutation aléatoire avec des taux de convergence arbitraires, ce qui donne en particulier une réponse positive à cette conjecture. Un résumé des résultats du Chapitre 2 est donné dans la Section A.2.

Malgré la vaste littérature sur les systèmes à commutation en dimension infinie [7, 79, 92, 111, 124, 149], peu de travaux se sont intéressés à des systèmes à excitation persistante en dimension infinie [47, 91]. En particulier, [91] analyse la stabilité de systèmes du type (A.6) avec A un opérateur (typiquement non-borné) sur un espace de Hilbert H qui engendre un semi-groupe fortement continu de contractions, $B \in \mathcal{L}(\mathsf{U}, \mathsf{H})$ pour un certain espace de Hilbert U, $K = -B^*$, et α un signal à excitation persistante, montrant qu'une inégalité d'observabilité généralisée est suffisante pour la stabilité exponentielle de (A.6) et qu'une propriété de continuation unique est suffisante pour la convergence faible des solutions de (A.6) vers zéro. Des résultats de stabilité de (A.6) pour des signaux α satisfaisant d'autres conditions plus générales que celle d'excitation persistante (A.4), garantissant toujours une action suffisante du contrôle sur le système, sont aussi donnés dans [91, 92]. Par contre, plusieurs problèmes restent ouverts, notamment le cas des opérateurs de contrôle non-bornés, des semi-groupes qui ne sont pas des contractions, ou des dynamiques sur des espaces de Banach.

Motivé par ces problèmes ouverts en dimension infinie, le Chapitre 3 de cette thèse s'intéresse à la stabilité du système d'équations de transport linéaires

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(x) u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^{N} m_{ij} u_j(t, L_j), & t \geq 0,\ i \in [\![1, N]\!], \\ u_i(0, x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!], \end{cases} \tag{A.7}$$

où, pour $i \in [\![1, N]\!]$, χ_i est la fonction caractéristique d'un intervalle $[a_i, b_i] \subset [0, L_i]$ avec $a_i < b_i$, α_i est un signal à excitation persistante, et la matrice $M = (m_{ij})_{i,j \in [\![1,N]\!]} \in \mathcal{M}_N(\mathbb{R})$, qui détermine les conditions aux bords, est appelée *matrice de transmission*. Le résultat principal du Chapitre 3 est le Théorème 3.1, qui donne des conditions suffisantes pour la stabilité de (A.7) en termes de la matrice de transmission et de la rationalité des rapports de longueurs L_i/L_j pour $i, j \in [\![1, N]\!]$. Un résumé des résultats du Chapitre 3 est donné dans la Section A.3.

A.1.3 Systèmes d'équations aux dérivées partielles sur des réseaux

L'étude du système d'équations de transport traité dans le Chapitre 3 est motivé non seulement par l'étude des systèmes à excitation persistante en dimension infinie mais également par le fait qu'il s'agit d'un modèle très simple de *système multi-corps*. Il s'agit de systèmes où cordes, membranes ou plaques interconnectées sont décrites par des équations aux dérivées partielles sur plusieurs domaines couplés, qui ont été beaucoup étudiés depuis les années 1980 [4, 5, 119, 120, 137, 138]. Cette activité de recherche est motivée par les applications des systèmes multi-corps et les questions mathématiques intéressantes qu'ils soulèvent (voir, par exemple, [6, 110] et leurs références).

Un cas particulier très important, qui comprend le système d'équations de transport étudié dans le Chapitre 3, est celui des systèmes d'équations aux dérivées partielles sur des réseaux unidimensionnels [35, 63]. Il s'agit de systèmes d'EDPs sur des domaines unidimensionnels, chaque domaine étant identifié à une arête d'un graphe, les interactions entre les EDPs ayant lieu aux nœuds du graphe. Malgré la simplification provenant de la dynamique unidimensionnelle dans chaque arête, l'analyse de ces systèmes est loin d'être triviale à cause des interactions aux nœuds. Par exemple, [63, Corolaire 5.38] montre que la contrôlabilité approchée d'un système d'équations d'ondes sur un réseau étoilé, contrôlé par un contrôle de Dirichlet dans un de ses nœuds extérieurs et avec des conditions de Dirichlet homogènes sur les autres, est équivalente à l'irrationalité de tous les rapports de longueurs de deux arêtes non-contrôlées différentes. La topologie du réseau peut aussi avoir une influence sur le comportement du système, comme l'illustre [48, Théorème 5.16], où l'on montre qu'un système d'équations d'ondes sur un réseau amorti dans ses nœuds extérieurs est exponentiellement stable si et seulement si le réseau est un arbre et tous les nœuds extérieurs sont amortis sauf au plus un.

Plusieurs types de systèmes d'EDPs sur des réseaux ont été traités dans la littérature, comme les systèmes d'équations des poutres d'Euler–Bernoulli [8, 130, 157], d'équations d'ondes [2, 25, 62, 63, 139, 176], de lois de conservation [24, 145], ou d'équations de Schrödinger [26, 98]. Dans plusieurs cas, l'analyse n'est faite que pour certaines topologies de réseau, comme les réseaux étoilés (des réseaux avec un nœud central appartenant à toutes les arêtes) [62, 79] ou les arbres (des réseaux sans cycles) [2, 26, 98, 145], mais, malgré cette simplification topologique, ces systèmes présentent encore plusieurs phénomènes intéressants dus à la structure du réseau.

L'analyse du système (A.7) dans le Chapitre 3 est faite grâce à une formule explicite pour ses solutions, obtenue à travers la méthode des caractéristiques et un argument itératif. En effet, on remarque que toute solution régulière de (A.7) satisfait, pour $i \in [\![1, N]\!]$, $x \in [0, L_i]$, et $t \geq x$,

$$u_i(t, x) = u_i(t - x, 0) e^{-\int_0^x \alpha_i(t-s) \chi_i(x-s) ds}, \tag{A.8}$$

où $\chi_i \equiv 0$ pour $i \in [\![N_d + 1, N]\!]$, avec une formule similaire pour exprimer $u_i(t, x)$ en fonction de la condition initiale $u_{i,0}$ lorsque $0 \leq t < x$. En utilisant la troisième équation de (A.7), on obtient ainsi que, pour $t \geq L_{\max}$,

$$u_i(t, 0) = \sum_{j=1}^{N} m_{ij} u_j(t - L_j, 0) e^{-\int_0^{L_j} \alpha_j(t-s) \chi_j(L_j - s) ds}. \tag{A.9}$$

La formule explicite pour $u_i(t, 0)$ est obtenue en itérant (A.9) afin de remonter dans le temps et exprimer $u_i(t, 0)$ en fonction des conditions initiales $u_{j,0}$, $j \in [\![1, N]\!]$, la formule explicite pour $u_i(t, x)$ pouvant être obtenue à partir de celle-ci et de (A.8).

On remarque également que la fonction $v : \mathbb{R}_+ \to \mathbb{R}^N$ définie par $v(t) = (u_i(t, 0))_{i \in [\![1, N]\!]} \in \mathbb{R}^N$ satisfait

$$v(t) = \sum_{j=1}^{N} A_j(t) v(t - L_j), \tag{A.10}$$

où la matrice $A_j(t) \in \mathcal{M}_N(\mathbb{R})$ est définie par $A_j(t) = \left(a_{k\ell}^{(j)}(t)\right)_{k, \ell \in [\![1, N]\!]}$, avec $a_{k\ell}^{(j)}(t) = 0$ pour $\ell \neq j$ et $a_{kj}^{(j)}(t) = m_{kj} \exp(-\int_0^{L_j} \alpha_j(t-s) \chi_j(L_j - s) ds)$. L'équation (A.10) est appelée une équation aux différences. Motivé par le fait que d'autres systèmes d'EDPs hyperboliques sur des réseaux plus généraux que (A.7) peuvent également s'exprimer comme des équations aux différences du type (A.10) et que les techniques utilisées dans le Chapitre 3 peuvent aussi s'appliquer à des équations aux différences plus générales, les Chapitres 4 et 5 de cette thèse s'intéressent, respectivement, à l'analyse de la stabilité de (A.10) et ses conséquences pour des systèmes de transport et d'ondes sur des réseaux, et à l'étude de la contrôlabilité d'un système d'équations aux différences.

A.1.4 Équations aux différences

L'analyse des équations aux différences autonomes a été l'objet de plusieurs travaux de recherche depuis les années 1970 [14, 60, 64, 84, 94, 129] (voir aussi [86, Chapitre 9] et ses références) et jusqu'à nos jours [48, 87, 127, 132]. Il s'agit de l'étude des équations du type

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j), \qquad t \geq 0, \tag{A.11}$$

avec $x(t) \in \mathbb{C}^d$, et, pour $j \in [\![1, N]\!]$, $A_j \in \mathcal{M}_d(\mathbb{C})$ et $\Lambda_j > 0$. On considère comme état du système la fonction $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0)}$, qui évolue donc dans un espace de dimension infinie. Ce système peut être étudié dans plusieurs espaces fonctionnels différents, tels que les espaces de Lebesgue L^p, les espaces de Sobolev $W^{k,p}$, ou les espaces $\mathcal{C}^k$, avec possiblement des conditions de compatibilité à prescrire pour garantir la régularité voulue. Dans ce qui suit, nous considérons l'espace de fonctions continues $\mathcal{C}^0$.

La stabilité de (A.11) a été étudiée par des méthodes spectrales et en utilisant des transformées de Laplace, conduisant à des critères de stabilité à $A_1, \ldots, A_N$ et $\Lambda_1, \ldots, \Lambda_N$ fixés, comme ceux de [60, 86, 94]. Cependant, si l'on peut montrer assez aisément que cette stabilité est robuste par rapport à des perturbations dans les matrices $A_1, \dots, A_N$, la situation est assez différente par rapport à des perturbations sur les retards $\Lambda_1, \dots, \Lambda_N$, comme constaté dans [94, 129, 134], puisque des perturbations dans les retards peuvent changer drastiquement la stabilité de (A.11). Le résultat suivant, connu sous le nom de *critère de Hale–Silkowski*, donne un critère de stabilisation robuste par rapport à des perturbations sur les retards, mettant aussi en évidence un lien entre une telle robustesse et les relations d'irrationalité entre les retards.

Théorème A.4 [14, Théorème 5.2]**.** *Soient $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$. Les affirmations suivantes sont équivalentes.*

(a) *On a $\rho_{\mathrm{HS}}(A) < 1$, où*

$$\rho_{\mathrm{HS}}(A) = \max_{(\theta_1, \dots, \theta_N) \in [0, 2\pi]^N} \rho\left(\sum_{j=1}^N e^{i\theta_j} A_j\right). \tag{A.12}$$

(b) *Il existe $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$ rationnellement indépendants tels que (A.11) est uniformément asymptotiquement stable.*

(c) *Il existe $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$ et un voisinage V de Λ dans $(0, +\infty)^N$ tels que, pour tout $L = (L_1, \dots, L_N) \in V$, le système*

$$x(t) = \sum_{j=1}^N A_j x(t - L_j)$$

est uniformément asymptotiquement stable.

(d) *Pour tous $\Lambda_1, \dots, \Lambda_N \in (0, +\infty)$,* (A.11) *est uniformément asymptotiquement stable.*

La propriété (c) du Théorème A.4 est appelée *stabilité forte locale* de A.11, et (d) est appelée *stabilité forte globale*, ou tout simplement *stabilité forte*. L'équivalence entre (c) et (d) a été démontrée par Jack K. Hale dans [82], le théorème complet étant par la suite montré par Richard A. Silkowski dans [159]. Ce résultat a été généralisé à des situations où l'on a une structure de dépendance rationnelle des retards dans [132] et à des matrices A_j dépendantes du temps dans [48, 136]. Le cas des équations aux différences avec retards dépendants du temps a aussi été traité, par exemple, dans [15].

L'une des motivations pour l'étude des équations aux différences (A.11) est le lien entre celles-ci et les équations différentielles fonctionnelles neutres du type

$$\frac{d}{dt}\left[x(t) - \sum_{j=1}^N A_j x(t - \Lambda_j)\right] = L x_t, \tag{A.13}$$

où $x_t = x(t + \cdot)|_{[-r, 0]}$, $r \geq \max_{j \in \{1, \dots, N\}} \Lambda_j$, et $L : \mathcal{C}^0([-r, 0], \mathbb{C}^d) \to \mathbb{C}^d$ est un opérateur linéaire borné. Plusieurs résultats sur (A.13) peuvent être obtenus à partir de propriétés de (A.11). Par exemple, [86, Chapitre 9, Théorèmes 7.1 à 7.3] donnent des propriétés des orbites et des ensembles ω-limites des équations du type (A.13) valables lorsque l'équation aux différences associée (A.11) est fortement stable. Il y a aussi des liens entre les spectres des semi-groupes engendrés par (A.11) et (A.13), comme décrit dans [94], qui montre en particulier que la stabilité exponentielle de (A.11) est une condition nécessaire pour la stabilité exponentielle de (A.13).

Une autre motivation pour l'étude de (A.11) est le fait que plusieurs systèmes d'équations aux dérivées partielles hyperboliques peuvent s'écrire sous cette forme. Il s'agit d'une approche classique pour l'étude des EDPs hyperboliques, basée sur la méthode des caractéristiques, qui est utilisée dans la littérature depuis au moins les années 1960 [33, 34, 54, 74, 133, 160] et jusqu'à présent [48, 56, 57, 70, 79, 106].

Motivé par les résultats sur les équations aux différences autonomes et ses applications, le Chapitre 4 de cette thèse s'intéresse à l'équation aux différences non-autonome

$$x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \tag{A.14}$$

où $A_j : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})$ pour $j \in [\![1, N]\!]$. Grâce à une formule explicite pour ses solutions, qui généralise celle obtenue dans le Chapitre 3 pour (A.7), nous caractérisons le comportement exponentiel de (A.14) en termes de certains coefficients matriciels dépendants du temps, prenant également en compte la structure de dépendance rationnelle des retards $\Lambda_1, \dots, \Lambda_N$. Nos résultats généralisent le critère de Hale–Silkowski, y compris sa généralisation au cas de retards satisfaisant une structure de dépendance rationnelle considéré dans [132], caractérisant la stabilité exponentielle de (A.14) uniformément par rapport à $A = (A_1, \dots, A_N) \in L^\infty(\mathbb{R}, \mathfrak{B})$ pour un certain ensemble borné non-vide $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$. Grâce à des transformations d'EDPs hyperboliques à coefficients variables dans le temps en équations aux différences non-autonomes, nous appliquons nos résultats à l'analyse de stabilité de systèmes d'équations de transport et d'ondes sur des réseaux. Un résumé des résultats du Chapitre 4 est donné dans la Section A.4.

Les équations aux différences et les équations différentielles fonctionnelles neutres ont également été traitées dans la littérature du point de vue de la théorie du contrôle [87, 88, 140, 141, 143, 154], auquel cas on s'intéresse à des systèmes du type

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + B u(t) \tag{A.15}$$

ou à une équation différentielle fonctionnelle neutre contrôlée correspondante. Dans (A.15), $u(t) \in \mathbb{C}^m$ représente le contrôle et $B \in \mathcal{M}_{d,m}(\mathbb{C})$.

L'un des problèmes considérés dans la littérature pour ce type de système est la stabilisation forte par des retours d'état linéaires du type $u(t) = \sum_{j=1}^{N} K_j x(t - \Lambda_j)$. Il a été montré dans [87] que cette propriété est équivalente à l'existence, pour tous $L_1, \dots, L_N \in (0, +\infty)$, d'un $\varepsilon > 0$ pour lequel on a

$$\operatorname{rk} \begin{pmatrix} B & \mathrm{Id}_d - \sum_{j=1}^{N} A_j e^{-\lambda L_j} \end{pmatrix} = d$$

pour tout $\lambda \in \mathbb{C}$ avec $\operatorname{Re} \lambda \geq -\varepsilon$, une condition qui rappelle le critère de contrôlabilité de Hautus (voir, par exemple, [163, Lemme 3.3.7]). Une condition correspondante a également été donnée dans [87] pour les équations différentielles fonctionnelles neutres contrôlées.

La contrôlabilité de (A.15) est aussi un problème qui suscite beaucoup d'intérêt. Puisque la dynamique de (A.15) a lieu dans un espace de dimension infinie, plusieurs notions différentes de contrôlabilité peuvent être introduites, telles que la contrôlabilité exacte, approchée, spectrale, ou relative [51, 154].

La contrôlabilité relative consiste à contrôler uniquement l'état final $x(T) \in \mathbb{C}^d$, à la place de tout l'état $x_T = x(T + \cdot)|_{[-\Lambda_{\max}, 0)}$. Cette notion a été introduite pour étudier des systèmes avec un retard dans le terme de contrôle [19, 51, 105, 142], ayant ensuite été utilisée aussi pour des systèmes avec retard dans l'état [66, 148]. Le critère de contrôlabilité relative suivant a été donné dans [148] pour caractériser la contrôlabilité relative d'un cas particulier de (A.15).

Théorème A.5 [148, Théorème 4]. *Considérons l'équation aux différences contrôlée*

$$x(t) = x(t-1) + Ax(t-\Lambda) + Bu(t), \tag{A.16}$$

où $\Lambda \in \mathbb{N}^*$, $A \in \mathcal{M}_d(\mathbb{C})$, *et* $B \in \mathcal{M}_{d,m}(\mathbb{C})$. *Supposons que* $\operatorname{rk} B = m \in [\![1, d]\!]$. *Soit* $T \in \mathbb{N}$. *Alors les affirmations suivantes sont équivalents.*

(a) *Pour tout* $x_0 : [-\Lambda, 0) \to \mathbb{C}^d$ *et* $x_1 \in \mathbb{C}^d$, *il existe* $u : [0, T] \to \mathbb{C}^m$ *tel que l'unique solution* x *de* (A.16) *avec condition initiale* x_0 *et contrôle* u *satisfait* $x(T) = x_1$.

(b) *On a* $T \geq T_{\min}$ *et*
$$\operatorname{rk}\begin{pmatrix} B & AB & A^2B & \cdots & A^qB \end{pmatrix} = d,$$
où $T_{\min} = \left\lceil \frac{d}{m} - 1 \right\rceil \Lambda$ *et* $q = \frac{T_{\min}}{\Lambda} = \left\lceil \frac{d}{m} - 1 \right\rceil$.

D'autres notions de contrôlabilité pour (A.15) sont aussi traitées dans la littérature, par exemple dans [141, 154].

Le Chapitre 5 de cette thèse s'intéresse à la contrôlabilité de (A.15). On traite d'abord à la contrôlabilité relative, que l'on caractérise dans quelques espaces fonctionnels à l'aide d'une formule explicite pour les solutions de (A.15), généralisant celle du Chapitre 4 pour (A.14). On compare également la contrôlabilité relative pour des retards différents en fonction de leur structure de dépendance rationnelle, caractérisant aussi le temps minimal pour la contrôlabilité relative. Ces résultats contiennent le Théorème A.5 comme cas particulier. La contrôlabilité exacte et approchée de (A.15) dans l'espace $L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$ est aussi l'objet du Chapitre 5, qui les étudie d'abord pour des retards commensurables avant de les caractériser complètement pour des systèmes en dimension deux avec deux retards et un contrôle, sans l'hypothèse de commensurabilité des retards. Un résumé des résultats du Chapitre 5 est présenté dans la Section A.5.

A.2 Exposants de Lyapunov pour systèmes à commutation aléatoires en temps continu et applications à la stabilisabilité de systèmes de contrôle

A.2.1 Systèmes à commutation aléatoires en temps continu

Motivé par le problème de la stabilisation à taux arbitraire de systèmes à excitation persistante décrit dans la Section A.1.2 et l'étude des processus de Markov déterministes par morceaux [17, 27, 29, 65], le Chapitre 2 s'intéresse à l'analyse de la stabilité de systèmes linéaires à commutation du type

$$\dot{x}(t) = A_{\alpha(t)} x(t), \tag{A.17}$$

où $N, d \in \mathbb{N}^*$, $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{R})$, et le signal de commutation α appartient à la classe $\mathcal{P}$ des signaux à valeurs dans $\underline{N}$, continus à droite et constants par morceaux (avec un nombre fini de discontinuités sur chaque intervalle de temps borné). Pour $x_0 \in \mathbb{R}^d$ et $\alpha \in \mathcal{P}$, l'unique solution de (A.17) avec condition initiale x_0 et signal de commutation α est notée $\varphi_{\mathrm{c}}(\cdot; x_0, \alpha)$.

Soient $M \in \mathcal{M}_N(\mathbb{R})$ une matrice stochastique (i.e., $\sum_{j=1}^N M_{ij} = 1$ pour tout $i \in \underline{N}$ et $M_{ij} \geq 0$ pour tous $i, j \in \underline{N}$), $p \in \mathbb{R}^N$ un vecteur de probabilité (i.e., $p_i \geq 0$ pour tout $i \in \underline{N}$ et $\sum_{i=1}^N p_i =$

1), vu comme un vecteur ligne dans $\mathcal{M}_{1,N}(\mathbb{R})$ et satisfaisant $pM = p$, et $\mu_1, \dots, \mu_N$ des mesures de probabilité boréliennes sur $\mathbb{R}_+^*$ à espérance finie (étendues à $\mathbb{R}_+$ de façon canonique). On choisit l'état $i_1 \in \underline{N}$ aléatoire selon la loi de probabilité p. Pour $n \in \mathbb{N}^*$ et $i_1, \dots, i_n \in \underline{N}$, $t_1, \dots, t_{n-1} \in \mathbb{R}_+$ construits, on choisit le temps $t_n \in \mathbb{R}_+$ aléatoire selon la loi μ_{i_n} et l'état suivant $i_{n+1} \in \underline{N}$ est choisi selon la loi $(M_{i_n j})_{j \in \underline{N}}$ définie par la i_n-ème ligne de la matrice M.

Cette construction définit une loi de probabilité $\mathbb{P}$ dans l'espace $\Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*}$ muni de la tribu produit provenant de la tribu des boréliens dans $\mathbb{R}_+$ et de la tribu de toutes les parties de $\underline{N}$, déterminant aussi un processus de Markov à temps discret dans Ω (cf. Définition 2.1 et Proposition 2.2). À $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$, on associe le signal à commutation $\boldsymbol{\alpha}(\omega) \in \mathcal{P}$ qui vaut i_n sur l'intervalle $[\sum_{k=1}^{n-1} t_k, \sum_{k=1}^{n} t_k)$ pour tout $n \in \mathbb{N}^*$. L'application $\boldsymbol{\alpha} : \Omega \to \mathcal{P}$ est définie pour presque tout $\omega \in \Omega$ (cf. Proposition 2.4 et Définition 2.5) et l'on considère (A.17) comme un système à commutation aléatoire avec signaux de commutation $\boldsymbol{\alpha}(\omega)$. Pour $x_0 \in \mathbb{R}^d$ et presque tout $\omega \in \Omega$, on définit $\varphi_{\mathrm{rc}}(\cdot; x_0, \omega) = \varphi_{\mathrm{c}}(\cdot; x_0, \boldsymbol{\alpha}(\omega))$.

En général, φ_{rc} ne définit pas un système dynamique aléatoire (dans le sens de [13]) avec la translation en temps usuelle θ_t sur Ω satisfaisant $\boldsymbol{\alpha}(\theta_t \omega)(s) = \boldsymbol{\alpha}(\omega)(t+s)$ pour $t, s \in \mathbb{R}_+$ et presque tout $\omega \in \Omega$ (cf. Exemple 2.6), puisque cette translation en temps ne préserve pas la mesure $\mathbb{P}$. Cela empêche l'application du Théorème ergodique multiplicatif d'Oseledets [13, Théorème 3.4.1] pour pouvoir analyser le comportement asymptotique du système dynamique aléatoire (A.17). La stratégie pour surmonter cette difficulté est d'étudier un système associé en temps discret.

A.2.2 Système associé en temps discret et Théorème ergodique multiplicatif

On définit l'application $\varphi_{\mathrm{rd}} : \mathbb{N} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ pour $n \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, et presque tout $\omega = (i_n, t_n)_{n=1}^\infty$ par $\varphi_{\mathrm{rd}}(n; x_0, \omega) = \varphi_{\mathrm{rc}}\left(\sum_{k=1}^n t_k; x_0, \omega\right)$, avec la convention $t_0 = 0$. Il s'agit de ne regarder la dynamique de (A.17) qu'aux instants de temps correspondant aux commutations de $\boldsymbol{\alpha}(\omega)$ (certaines de ces commutations pouvant être triviales, puisque l'on peut avoir $i_{n+1} = i_n$ pour certains $n \in \mathbb{N}^*$). L'application φ_{rd} représente ainsi la solution du système à commutation en temps discret

$$x_{n+1} = e^{A_{i_{n+1}} t_{n+1}} x_n \tag{A.18}$$

avec condition initiale x_0.

Le principal avantage dans l'étude du système à temps discret (A.18) par rapport à (A.17) est que, puisque $(i_n, t_n)_{n=1}^\infty$ est un processus de Markov à temps discret et $pM = p$, la translation en temps discret $\theta : \Omega \to \Omega$ définie par $\theta((i_n, t_n)_{n=1}^\infty) = (i_{n+1}, t_{n+1})_{n=1}^\infty$ préserve la mesure $\mathbb{P}$, et ainsi φ_{rd} définit un système dynamique aléatoire à temps discret sur Ω, puisque cette application satisfait la propriété de cocycle

$$\varphi_{\mathrm{rd}}(n+m; x_0, \omega) = \varphi_{\mathrm{rd}}(n; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)), \quad \forall n, m \in \mathbb{N},\ \forall x_0 \in \mathbb{R}^d,\ \text{presque tout } \omega \in \Omega,$$

(cf. Propositions 2.13 et 2.19).

L'analyse du comportement asymptotique de (A.17) et (A.18) est faite à travers leurs *exposants de Lyapunov*, définis pour $x_0 \in \mathbb{R}^d \setminus \{0\}$ et presque tout $\omega \in \Omega$ par

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \limsup_{n \to \infty} \frac{1}{n} \log |\varphi_{\mathrm{rd}}(n; x_0, \omega)|,$$
$$\lambda_{\mathrm{rc}}(x_0, \omega) = \limsup_{t \to \infty} \frac{1}{t} \log |\varphi_{\mathrm{rc}}(t; x_0, \omega)|.$$

Le lien entre λ_{rd} et λ_{rc} est donné par le résultat suivant.

Théorème A.6 (Théorème 2.26)**.** *Pour tout $x_0 \in \mathbb{R}^d \setminus \{0\}$ et presque tout $\omega \in \Omega$, on a*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = m(\omega)\lambda_{\mathrm{rc}}(x_0, \omega),$$

où $m(\omega)$ est défini pour $\omega = (i_n, t_n)_{n=1}^{\infty}$ par

$$m(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} t_k,$$

et cette limite existe et appartient à $(0, +\infty)$ pour presque tout $\omega \in \Omega$.

La valeur de $m(\omega)$ peut être caractérisée en termes des lois de probabilité p et $\mu_1, \dots, \mu_N$ (cf. Proposition 2.25). En particulier, si θ est ergodique par rapport à la mesure $\mathbb{P}$ (cf. Proposition 2.17), m est constante presque partout et vaut

$$m = \sum_{i=1}^{N} p_i \int_{\mathbb{R}_+} t d\mu_i(t). \tag{A.19}$$

Puisque φ_{rd} définit un système dynamique aléatoire en temps discret, on peut caractériser le comportement asymptotique du système à temps discret (A.18) par le Théorème ergodique multiplicatif d'Oseledets et, grâce au Théorème A.6, cela peut être utilisé pour caractériser le comportement asymptotique du système à temps continu (A.17). On obtient ainsi le résultat suivant.

Théorème A.7 (Théorème 2.31)**.** *Il existe un sous-ensemble $\widehat{\Omega} \subset \Omega$ invariant par θ et de mesure totale tel que, pour tout $\omega = (i_n, t_n)_{n=1}^{\infty} \in \widehat{\Omega}$,*

(a) *la limite $\Psi(\omega) = \lim_{n\to\infty} \left(e^{A_{i_1}^{\mathrm{T}} t_1} \cdots e^{A_{i_n}^{\mathrm{T}} t_n} e^{A_{i_n} t_n} \cdots e^{A_{i_1} t_1}\right)^{1/2n}$ existe et est une matrice définie positive ;*

(b) *il existe un entier $q(\omega) \in \underline{d}$ et $q(\omega)$ sous-espaces vectoriels $V_1(\omega), \dots, V_{q(\omega)}(\omega)$ de dimensions respectives $d_1(\omega) > \cdots > d_{q(\omega)}(\omega)$ tels que*

$$V_{q(\omega)}(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d,$$

et $e^{A_{i_1} t_1} V_i(\omega) = V_i(\theta(\omega))$ pour tout $i \in \underline{q(\omega)}$;

(c) *pour tout $x_0 \in \mathbb{R}^d \setminus \{0\}$, les exposants de Lyapunov $\lambda_{\mathrm{rd}}(x_0, \omega)$ et $\lambda_{\mathrm{rc}}(x_0, \omega)$ sont des limites, i.e.,*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \lim_{n\to\infty} \frac{1}{n} \log\lvert\varphi_{\mathrm{rd}}(n; x_0, \omega)\rvert,$$
$$\lambda_{\mathrm{rc}}(x_0, \omega) = \lim_{t\to\infty} \frac{1}{t} \log\lvert\varphi_{\mathrm{rc}}(t; x_0, \omega)\rvert \ ;$$

(d) *il existe des nombres réels $\lambda_1^{\mathrm{d}}(\omega) > \cdots > \lambda_{q(\omega)}^{\mathrm{d}}(\omega)$ et $\lambda_1^{\mathrm{c}}(\omega) > \cdots > \lambda_{q(\omega)}^{\mathrm{c}}(\omega)$ tels que, pour tout $i \in \underline{q(\omega)}$,*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \lambda_i^{\mathrm{d}}(\omega) \iff \lambda_{\mathrm{rc}}(x_0, \omega) = \lambda_i^{\mathrm{c}}(\omega) \iff x_0 \in V_i(\omega) \setminus V_{i+1}(\omega),$$

où $V_{q(\omega)+1}(\omega) = \{0\}$;

(e) *les valeurs propres de* $\Psi(\omega)$ *sont* $e^{\lambda_1^{\mathrm{d}}(\omega)} > \cdots > e^{\lambda_{q(\omega)}^{\mathrm{d}}(\omega)}$;

(f) $q(\theta(\omega)) = q(\omega)$ *et, pour* $i \in \underline{q(\omega)}$, $d_i(\theta(\omega)) = d_i(\omega)$, $\lambda_i^{\mathrm{d}}(\theta(\omega)) = \lambda_i^{\mathrm{d}}(\omega)$, *et* $\lambda_i^{\mathrm{c}}(\theta(\omega)) = \lambda_i^{\mathrm{c}}(\omega)$;

(g) *si* θ *est ergodique,* q *est constante sur* $\widehat{\Omega}$, *ainsi que* d_i, λ_i^{d}, *et* λ_i^{c} *pour* $i \in \underline{q}$.

Les résultats qui concernent le système à temps discret (A.18) sont obtenus directement par une application du Théorème ergodique multiplicatif d'Oseledets à ce système, et ceux pour le système à temps continu (A.17) proviennent du Théorème A.6. Ainsi, même si φ_{rc} ne définit pas un système dynamique aléatoire en temps continu en général, les conclusions du Théorème ergodique multiplicatif d'Oseledets restent vraies pour ce système.

Dans le Chapitre 2, on caractérise également les exposants de Lyapunov maximaux λ_1^{d} et λ_1^{c}, que l'on note respectivement $\lambda_{\max}^{\mathrm{d}}$ et $\lambda_{\max}^{\mathrm{c}}$.

Corolaire A.8 (Corolaire 2.35). *Supposons que* θ *est ergodique. Alors* $\lambda_{\max}^{\mathrm{c}}$ *et* $\lambda_{\max}^{\mathrm{d}}$ *sont constants presque partout sur* Ω *et satisfont*

$$\begin{aligned} \lambda_{\max}^{\mathrm{d}} &\leq \inf_{n\in\mathbb{N}^*} \frac{1}{n} \int_\Omega \log\left|e^{A_{i_n}t_n}\cdots e^{A_{i_1}t_1}\right| d\mathbb{P}((i_k,t_k)_{k=1}^\infty), \\ \lambda_{\max}^{\mathrm{c}} &= \frac{\lambda_{\max}^{\mathrm{d}}}{m}, \end{aligned} \tag{A.20}$$

où m *est donné par* (A.19). *En particulier, si*

$$\textit{il existe } n \in \mathbb{N}^* \textit{ tel que } \int_\Omega \log\left|e^{A_{i_n}t_n}\cdots e^{A_{i_1}t_1}\right| d\mathbb{P}((i_k,t_k)_{k=1}^\infty) < 0, \tag{A.21}$$

alors les systèmes (A.17) *et* (A.18) *sont presque surement exponentiellement stables.*

Si en plus il existe $r > 1$ *tel que* $\int_{\mathbb{R}_+} t^r d\mu_i(t) < \infty$ *pour tout* $i \in \underline{N}$, *alors* (A.20) *est une égalité et* (A.21) *est une condition nécessaire et suffisante pour la stabilité exponentielle presque sure de* (A.17) *et pour celle de* (A.18).

A.2.3 Application à la stabilisation de systèmes de contrôle

Le Corolaire A.8 est utilisé, dans la Section 2.6, pour étudier la stabilisation par retour d'état linéaire du système de contrôle à commutation

$$\dot{x}(t) = Ax(t) + B_{\alpha(t)}u_{\alpha(t)}(t), \tag{A.22}$$

où $x(t) \in \mathbb{R}^d$, $A \in \mathcal{M}_d(\mathbb{R})$, $\alpha : \mathbb{R}_+ \to \underline{N}$ est un signal de commutation aléatoire comme avant, et, pour $j \in \underline{N}$, $u_j(t) \in \mathbb{R}^{m_j}$ pour un entier $m_j \in \mathbb{N}$ et $B_j \in \mathcal{M}_{d,m_j}(\mathbb{R})$. Cette étude est motivée par l'analyse de la stabilisation des systèmes à excitation persistante présentée dans la Section A.1.2, le système (A.22) pouvant s'écrire sous la forme (A.3) dans le cas particulier où $N = 2$, $B_1 = B$, $B_2 = 0$, $u_1 = u$, et que l'on considère que α prend ses valeurs dans $\{0,1\}$ à la place de $\{1,2\}$. Pour simplifier l'étude, on suppose que la matrice M est irréductible, et ainsi le vecteur de probabilité p satisfaisant $pM = p$ est unique et θ est ergodique par rapport à la mesure $\mathbb{P}$. Le résultat de stabilisation obtenu à partir du Corolaire A.8 est le suivant.

Théorème A.9 (Théorème 2.36). *Soient* $A \in \mathcal{M}_d(\mathbb{R})$ *et, pour* $j \in \underline{N}$, $B_j \in \mathcal{M}_{d,m_j}(\mathbb{R})$ *pour un certain* $m_j \in \mathbb{N}^*$ *et* $\mathsf{V}_j = \operatorname{Ran} \mathcal{C}(A, B_j)$. *Supposons que* $\mathsf{V}_1 \oplus \cdots \oplus \mathsf{V}_N = \mathbb{R}^d$. *Alors, pour tout* $\lambda \in \mathbb{R}$, *il existe des matrices* $K_j \in \mathcal{M}_{m_j,d}(\mathbb{R})$, $j \in \underline{N}$, *telles que l'exposant de Lyapunov maximal* $\lambda_{\max}^{\mathrm{c}}$ *du système à commutation aléatoire à boucle fermée*

$$\dot{x}(t) = (A + B_{\boldsymbol{\alpha}(\omega)(t)}K_{\boldsymbol{\alpha}(\omega)(t)})x(t)$$

satisfait $\lambda_{\max}^{\mathrm{c}}(\omega) \leq \lambda$ *pour presque tout* $\omega \in \Omega$.

La démonstration de ce résultat repose sur le Corolaire A.8 et sur le fait que, pour une paire de matrices $(A, B) \in \mathcal{M}_d(\mathbb{R}) \times \mathcal{M}_{d,m}(\mathbb{R})$ contrôlable et tout $\gamma \geq 1$, il existe $K \in \mathcal{M}_{m,d}(\mathbb{R})$ telle que $\left|e^{(A+BK)t}\right| \leq C\gamma^L e^{-\gamma t}$, avec $L \in \mathbb{N}$ ne dépendant que de la dimension d et $C > 0$ ne dépendant que de A, B, et d [42, Proposition 2.1]. En particulier, il montre que, dans le cas du système (A.3) avec α un signal de commutation aléatoire selon le modèle précédent et à valeurs dans $\{0, 1\}$, on peut obtenir une stabilisation presque sure à taux de convergence arbitraire, ce qui est en contraste avec les systèmes à excitation persistante déterministes (A.5). Cela confirme l'intuition que les signaux de commutation déstabilisant (A.6) dans le cadre de la Proposition A.3(b) sont très particuliers et correspondent à un ensemble de mesure nulle. Un lien plus précis entre (A.22) et (A.5) est établi dans la Remarque 2.38.

A.3 Équations de transport avec amortissement persistant sur un réseau

Le Chapitre 3 de cette thèse s'intéresse au système d'équations de transport

$$\begin{cases} \partial_t u_i(t,x) + \partial_x u_i(t,x) + \alpha_i(t)\chi_i(x)u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,x) + \partial_x u_i(t,x) = 0, & t \geq 0,\ x \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^{N} m_{ij} u_j(t, L_j), & t \geq 0,\ i \in [\![1, N]\!], \\ u_i(0,x) = u_{i,0}(x), & x \in [0, L_i],\ i \in [\![1, N]\!], \end{cases} \tag{A.23}$$

où $N_d \in \mathbb{N}$ est le nombre d'équations de transport amorties et, pour $i \in [\![1, N_d]\!]$, χ_i est la fonction caractéristique d'un intervalle $[a_i, b_i] \subset [0, L_i]$ avec $a_i < b_i$ et α_i est un signal à excitation persistante dans $\mathcal{G}(T, \mu)$ pour certaines constantes $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$. La matrice $M = (m_{ij})_{i,j\in[\![1,N]\!]} \in \mathcal{M}_N(\mathbb{R})$ détermine les conditions aux bords et est appelée matrice de transmission. Ce système peut être vu comme un système défini sur un graphe, avec un seul nœud central et N arêtes orientées reliant ce nœud à lui-même (cf. Figure 3.1). Son étude est motivé d'une part par l'analyse de systèmes à excitation persistante en dimension infinie, introduite dans la Section A.1.2, et, d'autre part, par l'analyse de systèmes d'EDPs sur des réseaux, introduite dans la Section A.1.3.

Le résultat principal du Chapitre 3 est le théorème suivant.

Théorème A.10 (Théorème 3.1)**.** *Supposons que $N \geq 2$, $N_d \geq 1$, $|M|_{\ell^1} \leq 1$, $m_{ij} \neq 0$ pour tous $i, j \in [\![1, N]\!]$, et qu'il existe $i_*, j_* \in [\![1, N]\!]$ tels que $L_{i_*}/L_{j_*} \notin \mathbb{Q}$. Alors, pour tous $T, \mu \in \mathbb{R}_+^*$ avec $T \geq \mu$, il existe $C, \gamma > 0$ tels que, pour tout $p \in [1, +\infty]$, toute condition initiale $u_{i,0} \in L^p(0, L_i)$, $i \in [\![1, N]\!]$, et tout choix de signaux $\alpha_i \in \mathcal{G}(T, \mu)$, $i \in [\![1, N_d]\!]$, la solution correspondante de (A.23) satisfait*

$$\sum_{i=1}^{N} \|u_i(t)\|_{L^p(0,L_i)} \leq C e^{-\gamma t} \sum_{i=1}^{N} \left\|u_{i,0}\right\|_{L^p(0,L_i)}, \qquad \forall t \in \mathbb{R}_+.$$

A.3.1 Existence et unicité des solutions

Avant de montrer le Théorème A.10, le Chapitre 3 montre l'existence et l'unicité des solutions de (A.23) dans l'espace de Banach $\mathsf{X}_p = \prod_{i=1}^{N} L^p(0, L_i)$ pour $p \in [1, +\infty)$ (le cas $p = +\infty$

est traité séparément dans la Remarque 3.26). Cela est fait en écrivant (A.23) sous la forme

$$\begin{cases} \dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t) B_i z(t), \\ z(0) = z_0, \end{cases}$$

où $z_0 = (u_{1,0}, \dots, u_{N,0})$, $z(t) = (u_1(t,\cdot), \dots, u_N(t,\cdot))$, l'opérateur $A : D(A) \subset \mathsf{X}_p \to \mathsf{X}_p$ est défini par

$$D(A) = \left\{ (u_1, \dots, u_N) \in \prod_{i=1}^{N} W^{1,p}(0, L_i) \,\middle|\, \forall i \in [\![1, N]\!],\ u_i(0) = \sum_{j=1}^{N} m_{ij} u_j(L_j) \right\},$$

$$A(u_1, \dots, u_N) = \left(-\frac{du_1}{dx}, \dots, -\frac{du_N}{dx} \right),$$

et, pour $i \in [\![1, N_d]\!]$, $B_i \in \mathcal{L}(\mathsf{X}_p)$ est défini par

$$B_i(u_1, \dots, u_N) = (0, \dots, 0, -\chi_i u_i, 0, \dots, 0),$$

avec le terme $-\chi_i u_i$ dans la i-ème composante. L'existence et l'unicité des solutions ont lieu dans le sens suivant.

Théorème A.11 (Théorème 3.5)**.** *Soient $p \in [1, +\infty)$ et $\alpha_i \in L^\infty(\mathbb{R}, [0, 1])$ pour $i \in [\![1, N_d]\!]$. Il existe une unique famille d'évolution $\{T(t,s)\}_{t \geq s \geq 0}$ d'opérateurs bornés dans X_p telle que, pour tous $s \geq 0$ et $z_0 \in D(A)$, $t \mapsto z(t) = T(t,s)z_0$ est l'unique fonction continue satisfaisant $z(s) = z_0$, $z(t) \in D(A)$ pour tout $t \geq s$, z est dérivable pour presque tout $t \geq s$, $\dot{z} \in L^\infty_{\mathrm{loc}}([s, +\infty), \mathsf{X}_p)$, et $\dot{z}(t) = Az(t) + \sum_{i=1}^{N_d} \alpha_i(t) B_i z(t)$ pour presque tout $t \geq s$.*

Ce théorème est montré dans l'Appendice 3.A, où l'on rappelle également la définition d'une famille d'évolution. On considère également la fonction continue $t \mapsto T(t,s)z_0$ comme une solution de (A.23) même dans le cas où $z_0 \in \mathsf{X}_p \setminus D(A)$.

A.3.2 Formule explicite

Après une discussion sur l'origine et l'importance des hypothèses du Théorème A.10, le Chapitre 3 établit, dans la Section 3.3, une formule explicite pour la solution de (A.23) en termes des conditions initiales $u_{i,0}$, $i \in [\![1, N]\!]$, et de certains coefficients obtenus à partir de la matrice M et des signaux α_i, $i \in [\![1, N_d]\!]$. Cette formule est montrée d'abord dans le cas d'un système sans les termes d'amortissement (cf. Théorème 3.15), où les notations sont plus simples, avant de passer au cas général, dont l'énoncé est le suivant.

Théorème A.12 (Théorème 3.18)**.** *Soit $(u_{1,0}, \dots, u_{N,0}) \in D(A)$. La solution correspondante $(u_1, \dots, u_N)$ de (A.23) satisfait, pour $i \in [\![1, N]\!]$,*

$$u_i(t,x) = \begin{cases} u_{i,0}(x-t) \exp\left(-\int_{[0,t] \cap [t-x+a_i, t-x+b_i]} \alpha_i(s) ds \right), & \text{si } 0 \leq t \leq x, \\ u_i(t-x, 0) \exp\left(-\int_{[0,t] \cap [t-x+a_i, t-x+b_i]} \alpha_i(s) ds \right), & \text{si } t \geq x, \end{cases}$$

et, pour $t \geq 0$, $u_i(t, 0)$ est donné par

$$u_i(t,0) = \sum_{j=1}^{N} \sum_{\substack{\mathfrak{n} \in \mathfrak{N}_j \\ L(\mathfrak{n}) \leq t}} \vartheta^{(i)}_{j, \mathfrak{n} + \left\lfloor \frac{t - L(\mathfrak{n})}{L_j} \right\rfloor \mathbf{1}_j, L_j - \{t - L(\mathfrak{n})\}_{L_j}, t} u_{j,0}\left(L_j - \{t - L(\mathfrak{n})\}_{L_j} \right),$$

où $\mathfrak{N} = \mathbb{N}^N$, $\mathfrak{N}_j = \{\mathfrak{n} = (n_1, \ldots, n_N) \in \mathfrak{N} \mid n_j = 0\}$, $L(\mathfrak{n}) = \sum_{j=1}^N L_j n_j$ *pour* $\mathfrak{n} = (n_1, \ldots, n_N) \in \mathfrak{N}$, $\{\mathbf{1}_1, \ldots, \mathbf{1}_N\}$ *est la base canonique de* $\mathbb{R}^N$*, et les coefficients* $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$ *sont définis, pour* $i, j \in [\![1, N]\!]$, $\mathfrak{n} \in \mathfrak{N}$, $x \in [0, L_j]$ *et* $t \in \mathbb{R}$*, par*

$$\vartheta_{j,\mathfrak{n},x,t}^{(i)} = \varepsilon_{j,\mathfrak{n},x,t} \vartheta_{j,\mathfrak{n},L_j,t}^{(i)}, \tag{A.24}$$

avec

$$\varepsilon_{j,\mathfrak{n},x,t} = \exp\left(-\int_{I_{j,\mathfrak{n},x,t}} \alpha_j(s) ds\right), \tag{A.25}$$

où $I_{j,\mathfrak{n},x,t} = [t - L(\mathfrak{n}) - L_j + \max(x, a_j), t - L(\mathfrak{n}) - L_j + b_j]$*, et*

$$\vartheta_{j,\mathbf{0},L_j,t}^{(i)} = m_{ij},$$
$$\vartheta_{j,\mathfrak{n},L_j,t}^{(i)} = \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} m_{kj} \vartheta_{k,\mathfrak{n}-\mathbf{1}_k,0,t}^{(i)}.$$

A.3.3 Idée de la démonstration du Théorème A.11

L'idée principale de la preuve du Théorème A.11 est d'étudier le comportement asymptotique des solutions de (A.23) par le comportement asymptotique des coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$ lorsque $|\mathfrak{n}|_1 \to +\infty$. En effet, la Proposition 3.24 utilise la formule explicite du Théorème A.12 pour montrer que la convergence exponentielle de ces coefficients lorsque $|\mathfrak{n}|_1 \to +\infty$ implique celle des solutions lorsque $t \to +\infty$. Une fois cela établi, le Théorème A.11 est montré à travers une analyse des coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$. Cela est fait en décomposant $\mathfrak{N}$ en deux parties, à l'aide d'un paramètre $\rho \in (0, 1)$, comme suit.

Définition A.13 (Définition 3.31)**.** Pour $k \in [\![1, N]\!]$ et $\rho \in (0, 1)$, on définit

$$\mathfrak{N}_b(\rho, k) = \{\mathfrak{n} = (n_1, \ldots, n_N) \in \mathfrak{N} \mid n_k \leq \rho |\mathfrak{n}|_{\ell^1}\},$$
$$\mathfrak{N}_b(\rho) = \bigcup_{k=1}^{N} \mathfrak{N}_b(\rho, k), \qquad \mathfrak{N}_c(\rho) = \mathfrak{N} \setminus \mathfrak{N}_b(\rho).$$

L'ensemble $\mathfrak{N}_b(\rho)$ représente les points de $\mathfrak{N}_b$ qui ont une composante beaucoup plus petite que les autres. La décroissance exponentielle des coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$ dans $\mathfrak{N}_b(\rho)$ est montrée dans le Théorème 3.32, pour un certain $\rho \in (0, 1)$ suffisamment petit, par des arguments combinatoires (cf. Appendice 3.D), qui n'utilisent pas l'amortissement de l'équation mais uniquement le fait que la matrice M ne fait pas croitre la masse de la solution (dans le sens où $|M|_{\ell^1} \leq 1$) et mélange les composantes de la solution au point central (dans le sens où $m_{ij} \neq 0$ pour tous $i, j \in [\![1, N]\!]$).

La décroissance exponentielle des coefficients $\vartheta_{j,\mathfrak{n},x,t}^{(i)}$ est plus délicate à établir dans $\mathfrak{N}_c(\rho)$, puisque, à cause des signaux à excitation persistante α_i, $i \in [\![1, N_d]\!]$, qui peuvent être zéro sur certains intervalles de temps, on peu avoir pas ou peu de décroissance due aux termes $\varepsilon_{j,\mathfrak{n},x,t}$ définis dans (A.25) et apparaissant dans (A.24). L'idée principale est donc d'utiliser les hypothèses d'excitation persistante des α_i et d'irrationalité du rapport de longueurs $L_{i_*}/L_{j_*} \notin \mathbb{Q}$ pour certains $i_*, j_* \in [\![1, N]\!]$ pour garantir que $\varepsilon_{j,\mathfrak{n},x,t}$ donne une décroissance “suffisante” “assez souvent”, ce qui est établi dans le Lemme 3.38 (cf. aussi la Remarque 3.40). Grâce à ce résultat, on montre la décroissance exponentielle des coefficients dans $\mathfrak{N}_c(\rho)$ dans le Théorème 3.42, ce qui permet de conclure la démonstration du Théorème A.11.

A.4 Stabilité d'équations aux différences non-autonomes et applications au transport et à la propagation d'ondes sur des réseaux

Dans le Chapitre 4, cette thèse s'intéresse à la stabilité d'équations aux différences non-autonomes du type

$$\Sigma_\delta(\Lambda, A): \qquad u(t) = \sum_{j=1}^{N} A_j(t) u(t - \Lambda_j), \tag{A.26}$$

où $u(t) \in \mathbb{C}^d$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$, et $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, ainsi qu'à ses applications à des équations de transport et d'ondes sur des réseaux. Cette étude est motivée par le fait que plusieurs systèmes d'équations aux dérivées partielles hyperboliques à coefficients variables dans le temps, et notamment des systèmes d'EDPs hyperboliques sur des réseaux, peuvent s'écrire sous la forme (A.26), et aussi par le fait que les équations aux différences autonomes, i.e., avec $A = (A_1, \dots, A_N)$ constant, ont été beaucoup étudiés dans la littérature avec plusieurs résultats importants de stabilité, comme rappelé dans la Section A.1.4.

A.4.1 Équations aux différences

Le premier résultat du Chapitre 4 est la Proposition 4.2, qui montre l'existence et l'unicité des solutions de $\Sigma_\delta(\Lambda, A)$ dans l'espace de toutes les fonctions de $[-\Lambda_{\max}, +\infty)$ à valeurs dans $\mathbb{C}^d$. Lorsque $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, on a aussi existence et unicité de solutions pour $\Sigma_\delta(\Lambda, A)$ dans l'espace de Banach $\mathsf{X}_p^\delta = L^p([-\Lambda_{\max}, 0], \mathbb{C}^d)$ muni de sa norme usuelle $\|\cdot\|_p$ pour $p \in [1, +\infty]$ (cf. Remarque 4.4).

Dans la suite, on obtient une formule explicite pour les solutions de $\Sigma_\delta(\Lambda, A)$, dans l'esprit de celle du Théorème A.12 pour (A.23).

Lemme A.14 (Lemme 4.13). *Soient $\Lambda \in (\mathbb{R}_+^*)^N$, $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, et une condition initiale $u_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$. La solution correspondante $u : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ de $\Sigma_\delta(\Lambda, A)$ est donnée, pour $t \geq 0$, par*

$$u(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi^{\Lambda, A}_{\mathbf{n} - e_j, t} A_j(t - \Lambda \cdot \mathbf{n} + \Lambda_j) u_0(t - \Lambda \cdot \mathbf{n}), \tag{A.27}$$

où les coefficients $\Xi^{\Lambda, A}_{\mathbf{n}, t}$ sont définis, pour $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$, $\mathbf{n} \in \mathbb{Z}^N$ et $t \in \mathbb{R}$, par

$$\Xi^{\Lambda, A}_{\mathbf{n}, t} = \begin{cases} 0, & \text{si } \mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ \mathrm{Id}_d, & \text{si } \mathbf{n} = 0, \\ \displaystyle\sum_{k=1}^{N} A_k(t) \Xi^{\Lambda, A}_{\mathbf{n} - e_k, t - \Lambda_k}, & \text{si } \mathbf{n} \in \mathbb{N}^N \setminus \{0\}. \end{cases} \tag{A.28}$$

Afin d'étudier la stabilité de (A.26), il est utile de regrouper dans (A.27) les termes où la condition initiale u_0 est évaluée à un même instant de temps. Remarquons que u_0 est évalué dans un même instant de temps dans deux termes différents de (A.27), correspondant à des indices $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$, si et seulement si $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$. Cela indique que la structure de dépendance rationnelle de Λ joue un rôle important dans ce regroupement de termes. Pour $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (R_+^*)^N$, on définit ainsi

$$\begin{aligned} Z(\Lambda) &= \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\}, & \\ V(\Lambda) &= \{L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L)\}, \qquad & V_+(\Lambda) = V(\Lambda) \cap (\mathbb{R}_+^*)^N, \\ W(\Lambda) &= \{L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L)\}, \qquad & W_+(\Lambda) = W(\Lambda) \cap (\mathbb{R}_+^*)^N. \end{aligned} \tag{A.29}$$

L'ensemble $V(\Lambda)$ peut être vu comme l'ensemble des points de $\mathbb{R}^N$ qui sont au moins aussi rationnellement dépendants que Λ, $W(\Lambda)$ étant le sous-ensemble contenant les points qui ont exactement la même structure de dépendance rationnelle que Λ. On introduit également la définition suivante.

Définition A.15 (Définition 4.10)**.** Soit $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$. On partitionne les ensembles $[\![1, N]\!]$ et $\mathbb{Z}^N$ selon les relations d'équivalence $\sim$ et $\approx$ définies comme suit : $i \sim j$ si $\Lambda_i = \Lambda_j$ et $\mathbf{n} \approx \mathbf{n}'$ si $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$. On dénote par $[\cdot]$ les classes d'équivalence de $\sim$ et $\approx$, et on définit $\mathcal{J} = [\![1, N]\!]/\sim$ et $\mathcal{Z} = \mathbb{Z}^N/\approx$.

Pour $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $L \in V_+(\Lambda)$, $[\mathbf{n}] \in \mathcal{Z}$, $[i] \in \mathcal{J}$, et $t \in \mathbb{R}$, on définit

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}',t}^{L,A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \sum_{j \in [i]} A_j(t),$$

et

$$\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j).$$

Grâce à (A.29) et à la Définition A.15, la formule explicite du Lemme A.14 peut s'écrire sous la forme suivante.

Proposition A.16 (Proposition 4.14)**.** *Soient $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, et une condition initiale $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. La solution correspondante $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ de $\Sigma_\delta(L, A)$ est donnée, pour $t \geq 0$, par*

$$u(t) = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z} \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \Theta_{[\mathbf{n}],t}^{L,\Lambda,A} u_0(t - L \cdot \mathbf{n}). \tag{A.30}$$

Une fois la formule explicite (A.30) établie, l'objectif de la suite de la Section 4.2 est de l'utiliser pour caractériser le comportement exponentiel de $\Sigma_\delta(L, A)$. On cherche, en plus, à caractériser ce comportement uniformément par rapport à A dans une classe $\mathcal{A} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$. On suppose ici que la classe $\mathcal{A}$ est *uniformément localement bornée*, dans le sens où, pour tout $I \subset \mathbb{R}$ compact, $\sup_{A \in \mathcal{A}} \|A\|_{L^\infty(I, \mathcal{M}_d(\mathbb{C})^N)}$ est fini, et on note par $\Sigma_\delta(L, \mathcal{A})$ la famille de systèmes $\Sigma_\delta(L, A)$ pour $A \in \mathcal{A}$. On caractérise le comportement asymptotique de $\Sigma_\delta(L, \mathcal{A})$ à travers (A.30) en termes de son type exponentiel et de son exposant de Lyapunov maximal.

Définition A.17 (Définition 4.16)**.** Soit $L \in (\mathbb{R}_+^*)^N$.

(a) Pour $p \in [1, +\infty]$, on dit que $\Sigma_\delta(L, \mathcal{A})$ est de *type exponentiel* $\gamma \in \mathbb{R}$ dans X_p^δ si, pour tout $\varepsilon > 0$, il existe $K > 0$ tel que, pour tous $A \in \mathcal{A}$ et $u_0 \in \mathsf{X}_p^\delta$, la solution correspondante u de $\Sigma_\delta(L, A)$ satisfait, pour tout $t \geq 0$,

$$\|u_t\|_p \leq K e^{(\gamma + \varepsilon)t} \|u_0\|_p.$$

On dit que $\Sigma_\delta(L, \mathcal{A})$ est *exponentiellement stable* dans X_p^δ s'il est de type exponentiel négatif.

(b) Soit $\Lambda \in (\mathbb{R}_+^*)^N$ tel que $L \in V_+(\Lambda)$. On dit que $\Sigma_\delta(L,\mathcal{A})$ est de (Θ,Λ)-*type exponentiel* $\gamma \in \mathbb{R}$ si, pour tout $\varepsilon > 0$, il existe $K > 0$ tel que, pour tous $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, et presque tout $t \in (L\cdot\mathbf{n} - L_{\max}, L\cdot\mathbf{n})$, on a

$$\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq K e^{(\gamma+\varepsilon)t}.$$

(c) Soit $\Lambda \in (\mathbb{R}_+^*)^N$ tel que $L \in V_+(\Lambda)$. On dit que $\Sigma_\delta(L,\mathcal{A})$ est de $(\widehat{\Xi},\Lambda)$-*type exponentiel* $\gamma \in \mathbb{R}$ si, pour tout $\varepsilon > 0$, il existe $K > 0$ tel que, pour tous $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, et presque tout $t \in \mathbb{R}$, on a

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq K e^{(\gamma+\varepsilon)L\cdot\mathbf{n}}.$$

(d) Pour $p \in [1,+\infty]$, l'*exposant de Lyapunov maximal de* $\Sigma_\delta(L,\mathcal{A})$ dans X_p^δ est défini par

$$\lambda_p(L,\mathcal{A}) = \limsup_{t\to+\infty} \sup_{A\in\mathcal{A}} \sup_{\substack{u_0\in\mathsf{X}_p^\delta\\ \|u_0\|_p=1}} \frac{\log\|u_t\|_p}{t},$$

où u est la solution de $\Sigma_\delta(L,A)$ avec condition initiale u_0.

Après remarquer que l'exposant de Lyapunov maximal $\lambda_p(L,\mathcal{A})$ est le plus petit type exponentiel de $\Sigma_\delta(L,\mathcal{A})$ dans X_p^δ (cf. Proposition 4.18), on établit le lien entre le type exponentiel et le (Θ,Λ)-type exponentiel pour $\Sigma_\delta(L,\mathcal{A})$.

Théorème A.18 (Théorème 4.22)**.** *Soient $\Lambda \in (\mathbb{R}_+^*)^N$ et $\mathcal{A}$ un ensemble uniformément localement borné. Pour tout $L \in V_+(\Lambda)$, si $\Sigma_\delta(L,\mathcal{A})$ est de (Θ,Λ)-type exponentiel γ alors, pour tout $p \in [1,+\infty]$, $\Sigma_\delta(L,\mathcal{A})$ est de type exponentiel γ dans X_p^δ. Réciproquement, pour tout $L \in W_+(\Lambda)$, s'il existe $p \in [1,+\infty]$ pour lequel $\Sigma_\delta(L,\mathcal{A})$ est de type exponentiel γ dans X_p^δ, alors $\Sigma_\delta(L,\mathcal{A})$ est de (Θ,Λ)-type exponentiel γ. Finalement, pour tous $L \in W_+(\Lambda)$ et $p \in [1,+\infty]$,*

$$\lambda_p(L,\mathcal{A}) = \limsup_{L\cdot\mathbf{n}\to+\infty} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in(L\cdot\mathbf{n}-L_{\max},L\cdot\mathbf{n})} \frac{\log\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{t}.$$

Ce théorème est montré à l'aide de la formule explicite (A.30). À l'image de la Proposition 3.24 du Chapitre 3, on montre qu'une borne exponentielle sur les coefficients $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}$ conduit à une borne exponentielle sur les solutions de $\Sigma_\delta(L,\mathcal{A})$ lorsque $L \in V_+(\Lambda)$, ce qui est un résultat attendu à cause de (A.30), mais aussi que la réciproque est vraie si $L \in W_+(\Lambda)$, car dans ce cas les termes $L\cdot\mathbf{n}$ dans (A.30) sont différents pour des classes d'équivalence $[\mathbf{n}]$ différentes.

Lorsque $\mathcal{A}$ est un ensemble invariant par translation, i.e., $A(t+\cdot) \in \mathcal{A}$ pour tout $A \in \mathcal{A}$ et $t \in \mathbb{R}$, on peut également comparer le (Θ,Λ)-type exponentiel et le $(\widehat{\Xi},\Lambda)$-type exponentiel.

Théorème A.19 (Théorème 4.26)**.** *Soient $\Lambda \in (\mathbb{R}_+^*)^N$ et $\mathcal{A}$ un sous-ensemble borné de $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ invariant par translation. Pour tout $L \in V_+(\Lambda)$, $\Sigma_\delta(L,\mathcal{A})$ est de $(\widehat{\Xi},\Lambda)$-type exponentiel γ si et seulement s'il est de (Θ,Λ)-type exponentiel γ.*

Par conséquent, pour tout $L \in V_+(\Lambda)$, si $\Sigma_\delta(L,\mathcal{A})$ est de $(\widehat{\Xi},\Lambda)$-type exponentiel γ alors, pour tout $p \in [1,+\infty]$, $\Sigma_\delta(L,\mathcal{A})$ est de type exponentiel γ dans X_p^δ. Réciproquement, pour tout $L \in W_+(\Lambda)$, s'il existe $p \in [1,+\infty]$ tel que $\Sigma_\delta(L,\mathcal{A})$ est de type exponentiel γ dans X_p^δ, alors $\Sigma_\delta(L,\mathcal{A})$ est de $(\widehat{\Xi},\Lambda)$-type exponentiel γ. Finalement, pour tous $L \in W_+(\Lambda)$ et $p \in [1,+\infty]$,

$$\lambda_p(L,\mathcal{A}) = \limsup_{L\cdot\mathbf{n}\to+\infty} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{\log\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{L\cdot\mathbf{n}}.$$

L'avantage du Théorème A.19 par rapport au Théorème A.18 est qu'il est en général plus simple de calculer ou estimer les coefficients $\Xi_{\mathbf{n},t}^{L,A}$ ou $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$ que les coefficients $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}$, grâce à la formule de récurrence (A.28) (voir aussi (4.7) et (4.13)).

La dernière partie de la Section 4.2 s'intéresse au cas particulier où $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ pour un certain sous-ensemble borné non-vide $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$. Ce cas correspond à regarder (A.26) comme un système à commutation avec signaux de commutation arbitraires à valeurs dans $\mathfrak{B}$. Motivé par la formule explicite (4.14), on pose la définition suivante.

Définition A.20 (Définition 4.28)**.** Soient $\Lambda \in (\mathbb{R}_+^*)^N$ et $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ un ensemble borné non-vide. On définit

$$\mu(\Lambda, \mathfrak{B}) = \limsup_{\substack{x \to +\infty \\ x \in \mathcal{L}(\Lambda)}} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{pour } r \in \mathcal{L}_x(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\lvert\mathbf{n}\rvert_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right|^{\frac{1}{x}},$$

où $\mathcal{L}(\Lambda) = \{\Lambda \cdot \mathbf{n} \mid \mathbf{n} \in \mathbb{N}^N\}$, $\mathcal{L}_x(\Lambda) = \mathcal{L}(\Lambda) \cap [0, x)$ pour $x \in \mathbb{R}_+$, et, pour $\mathbf{n} \in \mathbb{N}^N$, $V_{\mathbf{n}} = \{v \in [\![1, N]\!]^{\lvert\mathbf{n}\rvert_1} \mid$ pour tout $k \in [\![1, N]\!]$, $\#\{j \in [\![1, \lvert\mathbf{n}\rvert_1]\!] \mid v_j = k\} = n_k\}$.

On établit, dans le Théorème 4.29, le lien entre $\mu(\Lambda, \mathfrak{B})$ et l'exposant de Lyapunov maximal $\lambda_p(L, \mathcal{A})$, ce qui conduit au critère de stabilité suivant.

Corolaire A.21 (Corolaire 4.31)**.** *Soient $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ un ensemble borné non-vide et $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. Les affirmations suivantes sont équivalentes.*

(a) $\mu(\Lambda, \mathfrak{B}) < 1$.

(b) $\Sigma_\delta(\Lambda, \mathcal{A})$ *est exponentiellement stable dans* X_p^δ *pour un certain* $p \in [1, +\infty]$.

(c) $\Sigma_\delta(L, \mathcal{A})$ *est exponentiellement stable dans* X_p^δ *pour tous* $L \in V_+(\Lambda)$ *et* $p \in [1, +\infty]$.

Le Corolaire A.21 généralise le critère de Hale–Silkowski, Théorème A.4, et le résultat correspondant de [132], aux équations aux différences non-autonomes à commutation arbitraire, puisque l'on montre que la stabilité exponentielle pour un certain $\Lambda \in (\mathbb{R}_+^*)^N$ et $p \in [1, +\infty]$ est équivalente à la stabilité exponentielle pour tout L au moins aussi rationnellement dépendant que Λ, dans le sens où $L \in V_+(\Lambda)$, et pour tout $p \in [1, +\infty]$. En plus, cette stabilité exponentielle est caractérisée par $\mu(\Lambda, \mathfrak{B}) < 1$, ce qui généralise la condition $\rho_{\mathrm{HS}}(A) < 1$ du Théorème A.4. Par contre, on n'a pas l'égalité entre $\mu(\Lambda, \{A\})$ et $\rho_{\mathrm{HS}}(A)$ lorsque les composantes de Λ sont rationnellement indépendantes, comme on pourrait s'attendre. On propose ainsi, dans la Définition 4.32, une autre quantité, $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$, qui généralise $\rho_{\mathrm{HS}}(A)$ (cf. Proposition 4.33), et pour laquelle on peut montrer un résultat similaire au Corolaire A.21, mais avec une hypothèse supplémentaire (cf. Corolaire 4.37).

A.4.2 Équations de transport

Dans la suite du Chapitre 4, les résultats présentés dans la Section A.4.1 pour les équations aux différences sont appliqués à des systèmes d'équations de transport. Pour $L = (L_1, \dots, L_N) \in (\mathbb{R}_+^*)^N$ et $M = (m_{ij})_{i,j \in [\![1,N]\!]} : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, on considère le système

$$\Sigma_\tau(L, M) : \begin{cases} \dfrac{\partial u_i}{\partial t}(t, x) + \dfrac{\partial u_i}{\partial x}(t, x) = 0, & i \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_i], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, L_j), & i \in [\![1, N]\!],\ t \in [0, +\infty), \end{cases} \tag{A.31}$$

où $u_i(\cdot, \cdot)$ prend ses valeurs dans $\mathbb{C}$ pour $i \in [\![1, N]\!]$.

Après avoir donné une définition de solution de $\Sigma_\tau(L, M)$ dans un sens faible en utilisant les caractéristiques (cf. Définition 4.38), on établit le lien entre ce système et le système d'équations aux différences $\Sigma_\delta(L, A)$ dans la Proposition 4.39, ce qui donne en particulier l'existence et l'unicité des solutions de $\Sigma_\tau(L, M)$.

Comme pour les équations aux différences, on dénote par $\Sigma_\tau(L, \mathcal{M})$ la famille de systèmes $\Sigma_\tau(L, M)$ pour $M \in \mathcal{M}$, où $\mathcal{M} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{R}))$ est un ensemble uniformément localement borné. On s'intéresse au comportement asymptotique de $\Sigma_\tau(L, M)$ uniformément par rapport à $M \in \mathcal{M}$.

Au lieu d'étudier le comportement de $\Sigma_\tau(L, \mathcal{M})$ dans l'espace $\mathsf{X}_p^\tau = \prod_{i=1}^N L^p([0, L_i], \mathbb{C})$ pour $p \in [1, +\infty]$, on s'intéresse aux sous-espaces $\mathsf{Y}_p(R)$ définis, pour $R = (\rho_{ij})_{i \in [\![1, r]\!], j \in [\![1, N]\!]} \in \mathcal{M}_{r,N}(\mathbb{C})$ et $r \in \mathbb{N}$, par

$$\mathsf{Y}_p(R) = \left\{ u = (u_1, \dots, u_N) \in \mathsf{X}_p^\tau \;\middle|\; \forall i \in [\![1, r]\!],\ \sum_{j=1}^N \rho_{ij} \int_0^{L_j} u_j(x) dx = 0 \right\}$$

(remarquons que X_p^τ est obtenu comme cas particulier en prenant $R = 0$). En effet, lorsque l'on traite le cas des systèmes d'équations d'ondes sur des réseaux dans la Section 4.4, on montre que ces systèmes peuvent s'écrire sous la forme $\Sigma_\tau(L, M)$ dans un espace du type $\mathsf{Y}_p(R)$, où chaque ligne de la matrice R représente un cycle ou un chemin entre deux nœuds extérieurs non-amortis du réseau. On caractérise d'abord les matrices R pour lesquelles $\mathsf{Y}_p(R)$ est invariant par le flot de $\Sigma_\tau(L, M)$ (cf. Proposition 4.42), et, pour $\mathcal{M} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$ uniformément localement borné, on dénote par $\operatorname{Inv}(\mathcal{M})$ l'ensemble de toutes les matrices $R \in \mathcal{M}_{r,N}(\mathbb{C})$ invariantes par le flot de $\Sigma_\tau(L, M)$ pour tout $M \in \mathcal{M}$.

On montre ensuite que les résultats de stabilité pour $\Sigma_\delta(L, \mathcal{A})$ présentés dans la Section A.4.1 peuvent être transposés aux systèmes du type (A.31) (cf. Théorème 4.47 et Corolaire 4.48; voir aussi la Définition 4.44). En particulier, lorsque $\mathcal{M} = L^\infty(\mathbb{R}, \mathfrak{B})$ pour un certain $\mathfrak{B} \subset \mathcal{M}_N(\mathbb{C})$ borné, on obtient comme conséquence du Corolaire A.21 le résultat suivant sur la stabilité exponentielle de la famille $\Sigma_\tau(L, \mathcal{M})$.

Corolaire A.22 (Corolaire 4.48). *Soient $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B} \subset \mathcal{M}_N(\mathbb{C})$ borné, et $\mathcal{M} = L^\infty(\mathbb{R}, \mathfrak{B})$. Les affirmations suivantes sont équivalentes.*

(a) $\Sigma_\tau(\Lambda, \mathcal{M})$ *est exponentiellement stable dans* $\mathsf{Y}_p(R)$ *pour un certain* $p \in [1, +\infty]$ *et* $R \in \operatorname{Inv}(\mathcal{M})$.

(b) $\Sigma_\tau(L, \mathcal{M})$ *est exponentiellement stable dans* $\mathsf{Y}_p(R)$ *pour tous* $L \in V_+(\Lambda)$, $p \in [1, +\infty]$, *et* $R \in \operatorname{Inv}(\mathcal{M})$.

Ainsi, la stabilité de $\Sigma_\tau(L, \mathcal{M})$ ne dépend pas de l'espace $\mathsf{Y}_p(R)$ dans lequel on considère les solutions, et, à l'image du critère de Hale–Silkowski, la stabilité pour un certain $\Lambda \in (\mathbb{R}_+^*)^N$ est équivalente à la stabilité pour tout $L \in (\mathbb{R}_+^*)^N$ au moins aussi rationnellement dépendant que Λ, dans le sens où $L \in V_+(\Lambda)$.

A.4.3 Équations d'ondes sur des réseaux

La dernière partie du Chapitre 4 transpose les résultats présentés dans la Section A.4.1 aux systèmes d'équations d'ondes sur des réseaux. On commence par un rappel des notations élémentaires pour les graphes et les réseaux.

Un *graphe* $\mathcal{G}$ est une paire $(\mathcal{V}, \mathcal{E})$, où $\mathcal{V}$ est un ensemble, dont les éléments sont appelés *nœuds*, et

$$\mathcal{E} \subset \{\{q, p\} \mid q, p \in \mathcal{V},\ q \neq p\}.$$

Les éléments de $\mathcal{E}$ sont appelés *arêtes*, et, pour $e = \{q, p\} \in \mathcal{E}$, les nœuds q, p sont appelés les *extremités* de e. Une *orientation* sur $\mathcal{G}$ est définie par deux fonctions $\alpha, \omega : \mathcal{E} \to \mathcal{V}$ telles que, pour tout $e \in \mathcal{E}$, $e = \{\alpha(e), \omega(e)\}$. Pour $q, p \in \mathcal{V}$, un *chemin* de q à p est un n-uplet $(q = q_1, \dots, q_n = p) \in \mathcal{V}^n$ où, pour tout $j \in [\![1, n-1]\!]$, $\{q_j, q_{j+1}\} \in \mathcal{E}$. L'entier positif n est appelé la *longueur* du chemin. Un chemin de longueur n dans $\mathcal{G}$ est dit *fermé* si $q_1 = q_n$; *simple* si toutes les arêtes $\{q_j, q_{j+1}\}$, $j \in [\![1, n-1]\!]$, sont différentes ; et *élémentaire* si les nœuds $q_1, \dots, q_n$ sont deux à deux différents, sauf possiblement pour la paire (q_1, q_n). Un chemin fermé élémentaire est appelé un *cycle*. Un graphe sans cycles est appelé un *arbre*. On dit qu'un graphe $\mathcal{G}$ est *connexe* si, pour tous $q, p \in \mathcal{V}$, il existe un chemin de q à p. On dit que $\mathcal{G}$ est *fini* si $\mathcal{V}$ est un ensemble fini. Pour tout $q \in \mathcal{V}$, on dénote par $\mathcal{E}_q$ l'ensemble des arêtes pour lesquelles q est une extrémité, i.e.,

$$\mathcal{E}_q = \{e \in \mathcal{E} \mid q \in e\}.$$

La cardinalité de $\mathcal{E}_q$ est notée n_q. On dit que $q \in \mathcal{V}$ est *extérieur* si $\mathcal{E}_q$ contient au plus un élément et *intérieur* sinon. On dénote par $\mathcal{V}_{\mathrm{ext}}$ et $\mathcal{V}_{\mathrm{int}}$ les ensembles de nœuds extérieurs et intérieurs, respectivement. On suppose dans ce qui suit que l'ensemble $\mathcal{V}_{\mathrm{ext}}$ contient au moins deux éléments, et l'on fixe un sous-ensemble non-vide $\mathcal{V}_{\mathrm{d}}$ de $\mathcal{V}_{\mathrm{ext}}$ tel que $\mathcal{V}_{\mathrm{u}} = \mathcal{V}_{\mathrm{ext}} \setminus \mathcal{V}_{\mathrm{d}}$ soit non-vide. Les nœuds de $\mathcal{V}_{\mathrm{d}}$ sont dits *amortis*, et ceux de $\mathcal{V}_{\mathrm{u}}$, *non-amortis*. On remarque que $\mathcal{V}$ est l'union disjointe $\mathcal{V} = \mathcal{V}_{\mathrm{int}} \cup \mathcal{V}_{\mathrm{u}} \cup \mathcal{V}_{\mathrm{d}}$.

Un *réseau* est une paire $(\mathcal{G}, L)$ où $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ est un graphe orienté et $L = (L_e)_{e \in \mathcal{E}}$ est un vecteur de nombres réels positifs, chaque L_e étant appelé la *longueur* de l'arête e. On dit qu'un réseau est *fini* (respectivement *connexe*) si le graphe $\mathcal{G}$ est fini (respectivement *connexe*). Si $e \in \mathcal{E}$ et $u : [0, L_e] \to \mathbb{C}$ est une fonction, on écrit $u(\alpha(e)) = u(0)$ et $u(\omega(e)) = u(L_e)$. Pour un chemin élémentaire $(q_1, \dots, q_n)$, sa *signature* $s : \mathcal{E} \to \{-1, 0, 1\}$ est définie par

$$s(e) = \begin{cases} 1, & \text{si } e = \{q_i, q_{i+1}\} \text{ pour un certain } i \in [\![1, n-1]\!] \text{ et } \alpha(e) = q_i, \\ -1, & \text{si } e = \{q_i, q_{i+1}\} \text{ pour un certain } i \in [\![1, n-1]\!] \text{ et } \alpha(e) = q_{i+1}, \\ 0, & \text{sinon.} \end{cases}$$

Les *dérivées normales* de u dans $\alpha(e)$ et $\omega(e)$ sont définies par $\frac{du}{dn_e}(\alpha(e)) = -\frac{du}{dx}(0)$ et $\frac{du}{dn_e}(\omega(e)) = \frac{du}{dx}(L_e)$.

Dans la suite, on ne considère que des réseaux finis connexes. Pour simplifier les notations, $\mathcal{E}$ est identifié à $[\![1, N]\!]$, où $N = \#\mathcal{E}$. Le système auquel on s'intéresse est

$$\Sigma_\omega(\mathcal{G}, L, \eta) : \begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(t, x) = \dfrac{\partial^2 u_j}{\partial x^2}(t, x), & j \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_j], \\ u_j(t, q) = u_k(t, q), & q \in \mathcal{V},\ j, k \in \mathcal{E}_q,\ t \in [0, +\infty), \\ \displaystyle\sum_{j \in \mathcal{E}_q} \dfrac{\partial u_j}{\partial n_j}(t, q) = 0, & q \in \mathcal{V}_{\mathrm{int}},\ t \in [0, +\infty), \\ \dfrac{\partial u_j}{\partial t}(t, q) = -\eta_q(t) \dfrac{\partial u_j}{\partial n_j}(t, q), & q \in \mathcal{V}_{\mathrm{d}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty), \\ u_j(t, q) = 0, & q \in \mathcal{V}_{\mathrm{u}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty), \end{cases} \tag{A.32}$$

où $u_j : [0, +\infty) \times [0, L_j] \to \mathbb{C}$ pour $j \in [\![1, N]\!]$. On suppose que la fonction η_q est positive ou nulle, déterminant l'amortissement au nœud $q \in \mathcal{V}_{\mathrm{d}}$, et on note $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$. On s'intéresse à la dynamique de $\Sigma_\omega(\mathcal{G}, L, \eta)$ dans l'espace $\mathsf{X}_p^\omega = W_0^{1,p}(\mathcal{G}, L) \times L^p(\mathcal{G}, L)$ pour $p \in [1, +\infty]$, où $L^p(\mathcal{G}, L) = \prod_{j=1}^N L^p([0, L_j], \mathbb{C})$ et $W_0^{1,p}(\mathcal{G}, L)$ est défini dans (4.41).

On montre d'abord l'équivalence entre $\Sigma_\omega(\mathcal{G}, L, \eta)$ et un système d'équations de transport $\Sigma_\tau(L, M)$ dans l'espace $\mathsf{Y}_p(R)$ pour un certain $R \in \mathcal{M}_{r,2N}(\mathbb{C})$, à travers l'opérateur de décomposition de d'Alembert (cf. Définition 4.51 et Proposition 4.58). Cette équivalence est utilisée ensuite, dans la Définition 4.60, pour définir les solutions de $\Sigma_\omega(\mathcal{G}, L, \eta)$ dans un sens faible, et l'existence et l'unicité de solutions est une conséquence du fait que l'espace $\mathsf{Y}_p(R)$ est invariant par le flot de l'équation de transport correspondante (cf. Proposition 4.61).

Comme dans les Sections A.4.1 et A.4.2, on s'intéresse à la famille de systèmes $\Sigma_\omega(\mathcal{G}, L, \eta)$ pour $\eta \in \mathcal{D}$, notée $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$, où $\mathcal{D}$ est un sous-ensemble de l'espace de fonctions mesurables $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$ à composantes positives ou nulles. Les résultats de la Section A.4.2 peuvent ainsi être transposés au cadre de (A.32), ce qui conduit en particulier au résultat suivant (voir la Définition 4.63 pour la définition de stabilité exponentielle dans ce contexte).

Corolaire A.23 (Corolaire 4.64)**.** *Soient $(\mathcal{G}, \Lambda)$ un réseau, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D} \subset (\mathbb{R}_+)^d$, et $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. Les affirmations suivantes sont équivalents.*

(a) *$\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ est exponentiellement stable dans X_p^ω pour un certain $p \in [1, +\infty]$.*

(b) *$\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ est exponentiellement stable dans X_p^ω pour tous $L \in V_+(\Lambda)$ et $p \in [1, +\infty]$.*

Comme dans le cas des Corolaires A.21 et A.22, le Corolaire A.23 montre que la stabilité pour un certain $\Lambda \in (\mathbb{R}_+^*)^N$ est équivalente à la stabilité pour tout $L \in (\mathbb{R}_+^*)^N$ au moins aussi rationnellement dépendant que Λ, dans le sens où $L \in V_+(\Lambda)$. Il permet, en plus, de montrer le critère de stabilité suivant.

Théorème A.24 (Théorème 4.65)**.** *Soient $(\mathcal{G}, \Lambda)$ un réseau, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D} \subset (\mathbb{R}_+)^d$ borné, et $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. Alors $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ est exponentiellement stable dans X_p^ω pour un certain $p \in [1, +\infty]$ si et seulement si $\mathcal{G}$ est un arbre, $\mathcal{V}_{\mathrm{u}}$ contient un seul nœud, et $\overline{\mathfrak{D}} \subset (\mathbb{R}_+^*)^d$.*

La partie "si" du Théorème A.24 peut être montrée par des méthodes classiques comme celles de [63, Chapitre 4, Section 4.1] (voir aussi [155]), en obtenant une inégalité d'observabilité à partir d'estimations d'énergie pour le système. D'autre part, le Corolaire A.23 permet de donner une démonstration simple de la partie "seulement si" du Théorème A.24. En effet, si $\mathcal{G}$ n'est pas un arbre, $\mathcal{V}_{\mathrm{u}}$ contient deux nœuds ou plus, ou $0 \in \overline{D}$, on montre que $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ n'est pas exponentiellement stable en construisant une solution périodique pour $\Sigma_\omega(\mathcal{G}, L, L^\infty(\mathbb{R}, \overline{D}))$ pour un certain $L \in V_+(\Lambda) \cap \mathbb{N}^N$. Le fait de prendre L à coefficients entiers permet de construire assez simplement une solution périodique pour ce système, et la conclusion pour $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ est alors une conséquence du Corolaire A.23.

A.5 Contrôlabilité d'équations aux différences linéaires

Motivé par les résultats sur les équations aux différences présentés dans la Section A.1.4, le Chapitre 5 de cette thèse s'intéresse à la contrôlabilité de l'équation aux différences

$$\Sigma(A, B, \Lambda): \qquad x(t) = \sum_{j=1}^N A_j x(t - \Lambda_j) + B u(t), \tag{A.33}$$

où $x(t) \in \mathbb{C}^d$ est l'état, $u(t) \in \mathbb{C}^m$ est le contrôle, $N, d, m \in \mathbb{N}^*$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, et $B \in \mathcal{M}_{d,m}(\mathbb{C})$. On aborde les questions de la contrôlabilité relative, exacte et approchée de (A.33).

On commence par une étude de l'existence et l'unicité des solutions de $\Sigma(A, B, \Lambda)$, qui sont d'abord établies, comme au Chapitre 4, dans l'ensemble de toutes les fonctions dans la Proposition 5.2, le cas de contrôle et état dans des espaces L^p et $\mathcal{C}^k$ étant traités dans les Remarques 5.3 et 5.4. En particulier, la régularité $\mathcal{C}^k$ des solutions n'est garantie que sous une condition de compatibilité entre la condition initiale et le contrôle à l'instant 0 (voir (5.4)). On dit qu'une condition initiale $x_0 \in \mathcal{C}^k([-\Lambda_{\max}, 0), \mathbb{C}^d)$ est $\mathcal{C}^k$-*admissible* lorsqu'il existe un contrôle u pour lequel la condition de compatibilité pour l'existence de solution C^k est satisfaite. À l'image du Lemme A.14, on donne une formule explicite pour les solutions de $\Sigma(A, B, \Lambda)$.

Proposition A.25 (Proposition 5.8)**.** *Soient $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, et $u : [0, T] \to \mathbb{C}^m$. La solution correspondante $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ de $\Sigma(A, B, \Lambda)$ est donnée, pour $t \in [0, T]$, par*

$$x(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t}} \Xi_{\mathbf{n}} B u(t - \Lambda \cdot \mathbf{n}), \tag{A.34}$$

où, pour $\mathbf{n} \in \mathbb{N}^N$, $\Xi_{\mathbf{n}}$ est défini comme dans (A.28)*.*

Remarquons que, différemment de (A.28), on note ici les coefficients matriciels simplement par $\Xi_{\mathbf{n}}$ à la place de $\Xi_{\mathbf{n},t}^{\Lambda,A}$ puisque ceux-ci ne dépendent ni de t ni de Λ, les matrices $A_1, \dots, A_N$ étant constantes. L'indice A est supprimé de la notation par souci de simplification. Comme dans la Section A.4.1, on regroupe dans cette formule les termes où le contrôle u est évalué au même instant de temps, à l'aide de la définition suivante.

Définition A.26 (Définition 5.11)**.** Soit $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$. On partitionne $\mathbb{N}^N$ selon la relation d'équivalence $\sim$ définie comme suit : on dit que $\mathbf{n} \sim \mathbf{n}'$ si $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$. On utilise $[\cdot]_\Lambda$ pour noter les classes d'équivalence de $\sim$ et on définit $\mathcal{N}_\Lambda = \mathbb{N}^N / \sim$. L'indice Λ est omis de la notation de $[\cdot]_\Lambda$ lorsque le vecteur des retards Λ en question est clair dans le contexte. On définit

$$\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}'}. \tag{A.35}$$

Avec cette définition, (A.34) s'écrit

$$x(t) = \sum_{\substack{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \\ -\Lambda_j \leq t - \Lambda \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}) + \sum_{\substack{[\mathbf{n}] \in \mathcal{N}_\Lambda \\ \Lambda \cdot \mathbf{n} \leq t}} \widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B u(t - \Lambda \cdot \mathbf{n}).$$

A.5.1 Contrôlabilité relative

La contrôlabilité relative de $\Sigma(A, B, \Lambda)$ en temps $T > 0$ consiste à savoir, étant donnés la condition initiale $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ et l'état final voulu $x_1 \in \mathbb{C}^d$, s'il est possible de trouver un contrôle $u : [0, T] \to \mathbb{C}^m$ tel que la solution x de $\Sigma(A, B, \Lambda)$ avec condition initiale x_0 et contrôle u satisfasse $x(T) = x_1$. Cette propriété peut aussi être posé dans d'autres espaces fonctionnels que l'ensemble de toutes les fonctions. On donne, dans le Chapitre 5, une caractérisation de la contrôlabilité relative dans les Théorèmes 5.12 et 5.13.

Théorème A.27 (Théorèmes 5.12 et 5.13)**.** *Soient $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$, $T > 0$, et $p \in [1, +\infty]$. On définit $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda}$ par* (A.35)*. Les quatre affirmations suivantes sont équivalentes.*

(a) *On a*

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B w \,\middle|\, [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d. \tag{A.36}$$

(b) *Pour tous* $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ *et* $x_1 \in \mathbb{C}^d$, *il existe* $u : [0, T] \to \mathbb{C}^m$ *tel que la solution* x *de* $\Sigma(A, B, \Lambda)$ *avec condition initiale* x_0 *et contrôle* u *satisfait* $x(T) = x_1$.

(c) *Il existe* $\varepsilon_0 > 0$ *tel que, pour tous* $\varepsilon \in (0, \varepsilon_0)$, $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$, *et* $x_1 : [0, \varepsilon] \to \mathbb{C}^d$, *il existe* $u : [0, T + \varepsilon] \to \mathbb{C}^m$ *tel que la solution* x *de* $\Sigma(A, B, \Lambda)$ *avec condition initiale* x_0 *et contrôle* u *satisfait* $x(T + \cdot)|_{[0,\varepsilon]} = x_1$.

(d) *Il existe* $\varepsilon_0 > 0$ *tel que, pour tous* $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$, *et* $x_1 \in L^p((0, \varepsilon), \mathbb{C}^d)$, *il existe* $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$ *tel que la solution* x *de* $\Sigma(A, B, \Lambda)$ *avec condition initiale* x_0 *et contrôle* u *satisfait* $x \in L^p((-\Lambda_{\max}, T + \varepsilon), \mathbb{C}^d)$ *et* $x(T + \cdot)|_{(0,\varepsilon)} = x_1$.

En plus, les trois affirmations suivantes sont équivalentes.

(e) *On a*

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} < T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d. \tag{A.37}$$

(f) *Pour tous* x_0 $\mathcal{C}^k$*-admissible pour* $\Sigma(A, B, \Lambda)$ *et* $x_1 \in \mathbb{C}^d$, *il existe* $u \in \mathcal{C}^k([0, T], \mathbb{C}^m)$ *tel que la solution* x *de* $\Sigma(A, B, \Lambda)$ *avec condition initiale* x_0 *et contrôle* u *satisfait* $x \in \mathcal{C}^k([-\Lambda_{\max}, T], \mathbb{C}^d)$ *et* $x(T) = x_1$.

(g) *Il existe* $\varepsilon_0 > 0$ *tel que, pour tous* $\varepsilon \in (0, \varepsilon_0)$, x_0 $\mathcal{C}^k$*-admissible pour* $\Sigma(A, B, \Lambda)$, *et* $x_1 \in \mathcal{C}^k([0, \varepsilon], \mathbb{C}^d)$, *il existe* $u \in \mathcal{C}^k([0, T + \varepsilon], \mathbb{C}^m)$ *tel que la solution* x *de* $\Sigma(A, B, \Lambda)$ *avec condition initiale* x_0 *et contrôle* u *satisfait* $x \in \mathcal{C}^k([-\Lambda_{\max}, T + \varepsilon], \mathbb{C}^d)$ *et* $x(T + \cdot)|_{[0,\varepsilon]} = x_1$.

Le Théorème A.27 montre ainsi que contrôler l'état final à un instant T est équivalent à le contrôler sur un petit intervalle de temps $[T, T + \varepsilon]$. La subtile différence entre (a) et (e) provient du fait que, pour contrôler dans l'espace $\mathcal{C}^k$, il faut aussi choisir un contrôle garantissant les conditions de compatibilité. On remarque également que les conditions (a) et (e) se réduisent au critère de Kalman lorsque $N = 1$, puisque, dans ce cas, $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} = \Xi_{\mathbf{n}} = A^n$ pour tout $\mathbf{n} = n \in \mathbb{N}$, où $A = A_1$.

Motivé par le Théorème A.27, on donne la définition suivante de contrôlabilité relative.

Définition A.28 (Définition 5.15)**.** Soient $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, $\Lambda \in (0, +\infty)^N$, et $T > 0$.

(a) On dit que $\Sigma(A, B, \Lambda)$ est *relativement contrôlable* en temps T si

$$\operatorname{Span}\left\{\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B w \;\middle|\; [\mathbf{n}] \in \mathcal{N}_\Lambda,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d.$$

(b) Si $\Sigma(A, B, \Lambda)$ est relativement contrôlable en temps $T > 0$, on définit son *temps minimal de contrôlabilité* $T_{\min}$ par $T_{\min} = \inf\{T > 0 \mid \Sigma(A, B, \Lambda)$ est relativement contrôlable en temps $T\}$.

La suite de l'étude de la contrôlabilité relative dans le Chapitre 5 consiste à étudier comment cette propriété dépend de la structure de dépendance rationnelle des composantes du vecteur de retards Λ et à caractériser le temps minimal de contrôlabilité $T_{\min}$. On définit un préordre sur l'ensemble des vecteurs de retards $(0, +\infty)^N$ qui détermine la structure de dépendance rationnelle de la façon suivante.

Définition A.29 (Définition 5.18)**.** Pour $\Lambda \in (0, +\infty)^N$, on définit $Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\}$. Pour $\Lambda, L \in (0, +\infty)^N$, on écrit $\Lambda \preccurlyeq L$ ou, de façon équivalente, $L \succcurlyeq \Lambda$, si $Z(\Lambda) \subset Z(L)$. On écrit $\Lambda \approx L$ si $\Lambda \preccurlyeq L$ et $L \preccurlyeq \Lambda$.

Ainsi, la relation $\Lambda \preccurlyeq L$ donne un sens à l'idée que L est "au moins aussi rationnellement dépendant" que Λ et correspond à dire que $L \in V_+(\Lambda)$ dans les notations de la Section A.4.1 introduites dans (A.29). Le premier résultat que l'on montre est que la contrôlabilité relative de $\Sigma(A, B, L)$ implique la contrôlabilité relative (en un temps différent) de $\Sigma(A, B, \Lambda)$ pour tout vecteur de retards $\Lambda \preccurlyeq L$ (cf. Théorème 5.20). La réciproque de ce résultat n'est pas vraie, comme illustré dans l'Exemple 5.21. On montre néanmoins que, pour tout $\Lambda \in (0, +\infty)^N$, il existe un vecteur de retards $L \succcurlyeq \Lambda$ à composantes commensurables et aussi proche que l'on veut de Λ de telle sorte que la contrôlabilité relative de $\Sigma(A, B, \Lambda)$ implique celle de $\Sigma(A, B, L)$ en même temps (cf. Théorème 5.22).

Concernant le temps minimal de contrôlabilité, on montre le résultat suivant, qui généralise le fait que, lorsque $N = 1$, le temps minimal de contrôlabilité $T_{\min}$ du système $x(t) = Ax(t-\Lambda)+Bu(t)$ satisfait, par le critère de Kalman et le Théorème de Cayley–Hamilton, $T_{\min} \leq (d-1)\Lambda$.

Théorème A.30 (Théorème 5.27)**.** *Soient $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$, $B \in \mathcal{M}_{d,m}(\mathbb{C})$, et $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$. S'il existe $T > 0$ tel que $\Sigma(A, B, \Lambda)$ est relativement contrôlable en temps T, alors sont temps minimal de contrôlabilité $T_{\min}$ satisfait $T_{\min} \leq (d-1)\Lambda_{\max}$.*

Le Théorème A.30 est montré d'abord pour un système avec retards commensurables (cf. Lemme 5.26), par une technique d'augmentation de l'état du système pour obtenir une équation aux différences à un seul retard, à laquelle on peut appliquer le critère de Kalman et le Théorème de Cayley–Hamilton. Les résultats comparant la contrôlabilité relative pour des retards différents en termes de leurs structures de dépendance rationnelle sont ensuite utilisés pour en déduire le cas général. Le Théorème 5.28 donne un autre critère de contrôlabilité relative qui peut permettre de calculer moins de coefficients $\widehat{\Xi}_{[\mathbf{n}]}^{\Lambda} B$, qui, dans le cas particulier des retards à composantes rationnellement indépendantes, montre qu'il suffit de calcules ces coefficients pour $\lvert\mathbf{n}\rvert_1 \leq d-1$ (cf. Corolaire 5.29).

A.5.2 Contrôlabilité exacte et approchée dans L^2

On considère ensuite le problème de la contrôlabilité exacte et approchée de (A.33) dans l'espace de Hilbert $\mathsf{X} = L^2((-\Lambda_{\max}, 0), \mathbb{C}^d)$ avec contrôles dans l'espace $\mathsf{Y}_T = L^2((0, T), \mathbb{C}^m)$ pour $T > 0$. Pour x une solution de (A.33) et $t \geq 0$, on écrit $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0)}$, et on remarque que, si $x_0 \in \mathsf{X}$ et $u \in \mathsf{Y}_T$, alors la solution x de (A.33) avec condition initiale x_0 et contrôle u satisfait $x_t \in \mathsf{X}$ pour tout $t \in [0, T]$.

Définition A.31 (Définition 5.32)**.** Soit $T \in (0, +\infty)$.

(a) On dit que (A.33) est *exactement contrôlable en temps T* si, pour tous $x_0, \overline{x} \in \mathsf{X}$, il existe $u \in \mathsf{Y}_T$ tel que la solution x de (A.33) avec condition initiale x_0 et contrôle u satisfait $x_T = \overline{x}$.

(b) On dit que (A.33) est *approximativement contrôlable en temps T* si, pour tous $x_0, \overline{x} \in \mathsf{X}$ et $\varepsilon > 0$, il existe $u \in \mathsf{Y}_T$ tel que la solution x de (A.33) avec condition initiale x_0 et contrôle u satisfait $\lVert x_T - \overline{x}\rVert_{\mathsf{X}} < \varepsilon$.

(c) On définit l'opérateur linéaire borné $E(T) : \mathsf{Y}_T \to \mathsf{X}$ par

$$(E(T)u)(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq T + t}} \Xi_{\mathbf{n}} B u(T + t - \Lambda \cdot \mathbf{n}).$$

Comme usuellement en théorie du contrôle, ces notions de contrôlabilité sont invariantes par changement de variables linéaire, changement d'échelle de temps, et retour d'état linéaire (cf. Lemme 5.33). En plus, la contrôlabilité exacte en temps T est équivalente à la surjectivité de $E(T)$ et à l'existence de $c > 0$ tel que $\|E(T)^* x\|_{\mathsf{Y}_T}^2 \geq c \|x\|_{\mathsf{X}}^2$ pout tout $x \in \mathsf{X}$; et la contrôlabilité approchée en temps T est équivalente à la densité de l'image de $E(T)$ et à l'injectivité de $E(T)^*$ (cf. Propositions 5.34 et 5.35). L'opérateur adjoint $E(T)^*$ peut être caractérisé par un calcul simple (cf. Lemme 5.36).

On traite d'abord la contrôlabilité exacte et approchée dans le cas où les retards $\Lambda_1, \dots, \Lambda_N$ sont commensurables par deux techniques différentes. On considère d'abord une augmentation de l'état du système (cf. Lemme 5.38), qui le transforme dans un système à un seul retard, pour lequel les contrôlabilités exacte et approchée peuvent être caractérisées par une condition du type Kalman (cf. Proposition 5.40). On obtient en particulier l'équivalence entre contrôlabilités exacte et approchée dans ce cas. La deuxième méthode consiste à étudier l'opérateur $E(T)$, que l'on représente par deux matrices C et E (cf. Lemme 5.46). On obtient à nouveau que les contrôlabilités exacte et approchée sont équivalentes et on les caractérise maintenant à l'aide du rang de la matrice C (cf. Proposition 5.47). Le lien entre ces deux résultats est montré dans le Théorème 5.49, qui établit que la matrice C est la matrice de Kalman du système augmenté du Lemme 5.38.

La suite du Chapitre 5 étudie la contrôlabilité sans l'hypothèse de commensurabilité sur les retards. Cette étude étant beaucoup plus délicate, on se restreint au cas $N = d = 2$ et $m = 1$, qui, malgré l'apparence simple, présente déjà plusieurs caractéristiques non-triviales. On s'intéresse ainsi au système

$$x(t) = A_1 x(t - \Lambda_1) + A_2 x(t - \Lambda_2) + B u(t), \tag{A.38}$$

où $x(t) \in \mathbb{C}^2$, $u(t) \in \mathbb{C}$, $A_1, A_2 \in \mathcal{M}_2(\mathbb{C})$, et $B \in \mathcal{M}_{2,1}(\mathbb{C})$, ce dernier ensemble étant identifié canoniquement à $\mathbb{C}^2$. Sans perte de généralité, on suppose $\Lambda_1 > \Lambda_2$. Le résultat principal obtenu pour le système (A.38) est le suivant.

Théorème A.32 (Théorème 5.51)**.** *Soient $T \in (0, +\infty)$ et $(\Lambda_1, \Lambda_2) \in (0, +\infty)^2$ avec $\Lambda_1 > \Lambda_2$.*

(a) *Si la paire (A_1, B) n'est pas contrôlable, alors (A.38) n'est ni exactement ni approximativement contrôlable en temps T.*

(b) *Si la paire (A_1, B) est contrôlable et (A_2, B) ne l'est pas, alors les affirmations suivantes sont équivalents.*

  (i) *Le système (A.38) est exactement contrôlable en temps T.*

  (ii) *Le système (A.38) est approximativement contrôlable en temps T.*

  (iii) $T \geq 2\Lambda_1$.

(c) *Si les paires (A_1, B) et (A_2, B) sont toutes les deux contrôlables, on fixe $Z \in \mathbb{C}^2 \setminus \operatorname{Span}\{B\}$ et on définit*

$$\beta = \frac{\det \mathcal{C}(A_1, B)}{\det \mathcal{C}(A_2, B)}, \qquad \alpha = \frac{\det\begin{pmatrix} B & (A_1 - \beta A_2) Z \end{pmatrix}}{\det\begin{pmatrix} B & Z \end{pmatrix}}. \tag{A.39}$$

Alors α ne dépend pas de Z. Soit $C \subset \mathbb{C}$ l'ensemble de toutes les valeurs complexes possibles de l'expression $\beta + \alpha^{1 - \frac{\Lambda_2}{\Lambda_1}}$.

  (i) *Le système (A.38) est exactement contrôlable en temps T si et seulement si $T \geq 2\Lambda_1$ et $0 \notin \overline{C}$.*

(ii) *Le système* (A.38) *est approximativement contrôlable en temps* T *si et seulement si* $T \geq 2\Lambda_1$ *et* $0 \notin C$.

Remarquons que α et β sont invariants par changement de variables linéaire et retour d'état linéaire (cf. Lemme 5.53). En utilisant des changements de variables, des retours d'état et des changements d'échelle de temps, on réduit la démonstration du Théorème A.32 à certaines formes canoniques (cf. Remarque 5.54). La démonstration des parties (a) et (b) est assez simple, reposant sur une étude de l'image de $E(T)$. Pour la partie (c), la non-contrôlabilité pour $T < 2\Lambda_1$ est démontrée en construisant une fonction dans le noyau de $E(T)^*$.

La partie la plus intéressante de la preuve du Théorème A.32 est le cas (c) lorsque $T \geq 2\Lambda_1$. On se réduit d'abord au cas $T = 2\Lambda_1$ (cf. Lemme 5.56), et on montre ensuite qu'il suffit d'étudier l'opérateur $S : L^2((-1,0),\mathbb{C}) \to L^2((-1,0),\mathbb{C})$ défini par

$$Sx(t) = \begin{cases} \overline{\beta}x(t) + x(t+L-1) & \text{if } -L < t < 0, \\ \overline{\beta}x(t) + \overline{\alpha}x(t+L) & \text{if } -1 < t < -L, \end{cases}$$

où $L = \frac{\Lambda_2}{\Lambda_1} \in (0,1)$. En effet, (A.38) est exactement contrôlable en temps $T = 2\Lambda_1$ si et seulement si S^* est surjectif ou, de façon équivalente, s'il existe $c > 0$ tel que $\|Sx\|_{L^2((-1,0),\mathbb{C})} \geq c\|x\|_{L^2((-1,0),\mathbb{C})}$ pour tout $x \in L^2((-1,0),\mathbb{C})$; et la contrôlabilité approchée de (A.38) en temps $T = 2\Lambda_1$ est équivalente à l'injectivité de S (cf. Lemme 5.58).

La démonstration de la partie (c)(ii) dans le cas $T \geq 2\Lambda_1$ est décomposée en deux parties, selon que $L = \frac{\Lambda_2}{\Lambda_1}$ est rationnel ou pas. Dans le premier cas, S est équivalent à une matrice M (dans le sens de (5.68)), dont le déterminant s'annule si et seulement si $0 \in C$, ce qui donne le résultat. Dans le deuxième cas, si $0 \in C$, on montre, en utilisant l'ergodicité de la translation par L modulo 1, que le noyau de S est l'ensemble des fonctions du type $x(t) = Ce^{\gamma t}$ pour $C \in \mathbb{C}$ et γ un logarithme de $\overline{\alpha}$. Si $0 \notin C$, on montre l'injectivité de S à l'aide d'un raisonnement sur la transformée de Fourier d'une fonction dans le noyau de S.

Pour la partie (c)(i) dans le cas $T \geq 2\Lambda_1$, on fait une étude précis de la matrice M qui est équivalente à l'opérateur S dans le sens de (5.68) lorsque L est rationnel, afin de montrer que, si $0 \notin \overline{C}$, la norme de son inverse reste bornée lorsque L s'approche d'un irrationnel, ce qui montre ainsi la contrôlabilité exacte de (A.38). Si $0 \in \overline{C}$, on montre que alors 0 est une valeur propre ou un point d'accumulation de valeurs propres de S, ce qui implique que l'on n'a pas la contrôlabilité exacte de (A.38).

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École doctorale de mathématiques Hadamard (EDMH)

Stabilité et stabilisation de systèmes linéaires à commutation en dimensions finie et infinie

Mots-clés. Systèmes à commutation, stabilité, stabilisation, excitation persistante, exposants de Lyapunov, commutation aléatoire, équation de transport, équation des ondes, équations aux différences, contrôlabilité, réseaux.

Résumé. Motivée par des travaux précédents sur la stabilisation de systèmes à excitation persistante, cette thèse s'intéresse à la stabilité et à la stabilisation de systèmes linéaires à commutation en dimensions finie et infinie. Après une introduction générale présentant les principales motivations et les résultats importants de la littérature, on aborde quatre sujets.

On commence par l'étude d'un système linéaire en dimension finie à commutation aléatoire. Le temps passé en chaque sous-système i est choisi selon une loi de probabilité ne dépendant que de i, les commutations entre sous-systèmes étant déterminées par une chaine de Markov discrète. On caractérise les exposants de Lyapunov en appliquant le Théorème ergodique multiplicatif d'Oseledets à un système associé en temps discret, et on donne une expression pour l'exposant de Lyapunov maximal. Ces résultats sont appliqués à un système de contrôle à commutation. Sous une hypothèse de contrôlabilité, on montre que ce système peut être stabilisé presque surement avec taux de convergence arbitraire, ce qui est en contraste avec les systèmes déterministes à excitation persistante.

On considère ensuite un système de N équations de transport avec amortissement interne à excitation persistante, couplées linéairement par le bord à travers une matrice M, ce qui peut être vu comme un système d'EDPs sur un réseau étoilé. On montre que, si l'activité de l'amortissement intermittent est déterminée par des signaux à excitation persistante, alors, sous des bonnes hypothèses sur M et sur la rationalité des rapports entre les longueurs des arêtes du réseau, ce système est exponentiellement stable, uniformément par rapport aux signaux à excitation persistante. Ce résultat est montré grâce à une formule explicite pour les solutions du système, qui permet de bien suivre les effets de l'amortissement intermittent.

Le sujet suivant que l'on considère est le comportement asymptotique d'équations aux différences non-autonomes. On obtient une formule explicite pour les solutions en termes des conditions initiales et de certains coefficients matriciels dépendants du temps, qui généralise la formule obtenue pour le système de N équations de transport. Le comportement asymptotique des solutions est caractérisé à travers les coefficients matriciels. Dans le cas d'équations aux différences à commutation arbitraire, on obtient un résultat de stabilité qui généralise le critère de Hale–Silkowski pour les systèmes autonomes. Grâce à des transformations classiques d'EDPs hyperboliques en équations aux différences, on applique ces résultats au transport et à la propagation d'ondes sur des réseaux.

Finalement, la formule explicite précédente est généralisée à une équation aux différences contrôlée, dont la contrôlabilité est alors analysée. La contrôlabilité relative est caractérisée à travers un critère algébrique sur les coefficients matriciels de la formule explicite, ce qui généralise le critère de Kalman. On compare également la contrôlabilité relative pour des retards différents en termes de leur structure de dépendance rationnelle, et on donne une borne sur le temps minimal de contrôlabilité. Pour des systèmes avec retards commensurables, on montre que la contrôlabilité exacte est équivalente à l'approchée et on donne un critère qui les caractérise. On analyse également la contrôlabilité exacte et approchée de systèmes en dimension 2 avec deux retards sans l'hypothèse de commensurabilité.

Stability and stabilization of linear switched systems in finite and infinite dimensions

Keywords. Switched systems, stability, stabilization, persistent excitation, Lyapunov exponents, random switching, transport equation, wave equation, difference equations, controllability, networks.

Abstract. Motivated by previous work on the stabilization of persistently excited systems, this thesis addresses stability and stabilization issues for linear switched systems in finite and infinite dimensions. After a general introduction presenting the main motivations and important results from the literature, we analyze four problems.

The first system we study is a linear finite-dimensional random switched system. The time spend on each subsystem i is chosen according to a probability law depending only on i, and the switches between subsystems are determined by a discrete Markov chain. We characterize the Lyapunov exponents by applying Oseledets' Multiplicative Ergodic Theorem to an associated discrete-time system, and provide an expression for the maximal Lyapunov exponent. These results are applied to a switched control system, showing that, under a controllability hypothesis, almost sure stabilization can be achieved with arbitrarily large decay rates, a situation in contrast to deterministic persistently excited systems.

We next consider a system of N transport equations with intermittent internal damping, linearly coupled by their boundary conditions through a matrix M, which can be seen as a system of PDEs on a star-shaped network. We prove that, if the activity of the intermittent damping terms is determined by persistently exciting signals, then, under suitable hypotheses on M and on the rationality of the ratios between the lengths of the network edges, such system is exponentially stable, uniformly with respect to the persistently exciting signals. The proof of this result is based on an explicit representation formula for the solutions of the system, which allows one to efficiently track down the effects of the intermittent damping.

The following topic we address is the asymptotic behavior of non-autonomous difference equations. We obtain an explicit representation formula for their solutions in terms of their initial conditions and some time-dependent matrix coefficients, which generalizes the one for the system of N transport equations. The asymptotic behavior of solutions is characterized in terms of the matrix coefficients. In the case of difference equations with arbitrary switching, we obtain a stability result which generalizes Hale–Silkowski criterion for autonomous systems. Using classical transformations of hyperbolic PDEs into difference equations, we apply our results to transport and wave propagation on networks.

Finally, we generalize the previous representation formula to a controlled difference equation, whose controllability is then analyzed. Relative controllability is characterized in terms of an algebraic property on the matrix coefficients from the explicit formula, generalizing Kalman criterion. We also compare the relative controllability for different delays in terms of their rational dependence structure, and provide a bound on the minimal controllability time. Exact and approximate controllability for systems with commensurable delays are characterized and proved to be equivalent. We also describe exact and approximate controllability for two-dimensional systems with two delays not necessarily commensurable.

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